OBSERVATIONS

ON

CONVERGING SERIES.

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CONVERGING SERIES,

OCCASIONED BY

MR. CLARKE's TRANSLATION

OF

MR. LORGNA's TREATISE

ON THE SAME SUBJECT.

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OBSERVATIONS

ON

CONVERGING SERIES.

IT having been aſſerted that the method exhibited in Mr. LORGNA's *Diſſertation on Converging Series* is not only entirely new, but much more general than any other which has hitherto appeared on the ſubject; I wiſh the intelligent analyſt to recollect the contents of the *Mathematical Diſſertations* of my deceaſed friend, Mr. THOMAS SIMPSON, wherein is pointed out, a very ready method of computing the ſums of a great number of ſuch ſeries, comprehending, at leaſt, all that can be done by the method exhibited with *ſo much oſtentation in the Tranſlation* of Mr. LORGNA's book.

Mr. SIMPSON having not given a full explanation of the method I allude to, by a formal and diffuſe application of it, in the ſummation of particular ſeries, I intend to explain it farther, and to endeavour to ſhew, that, as it is not leſs general, it is alſo not leſs elegant (indeed I eſteem it much more ſo) than Mr. LORGNA's *boaſted* method.

I.

It is ſhewn by Mr. Simpson, at page 87 of his *Diſſertations* above-mentioned, that the ſeries

$$(Q=)\ \frac{c\,d\,e\,(p)\times \overset{\prime}{P}x^{m}}{r\,s\,t\,(q+1)}+\frac{c+n.\ \ d+n.\ \ e+n.(p)\times \overset{\prime\prime}{P}x^{m+n}}{r+n.\ \ s+n.\ \ t+n(q+1)}$$

$$+\frac{c+2n.\ \ d+2n.\ \ e+2n(p)\times \overset{\prime\prime\prime}{P}x^{m+2n}}{r+2n.\ \ s+2n.\ \ t+2n(q+1)}\ \&c.$$

$$\text{will be } = A\overset{\prime}{S}+B\overset{\prime\prime}{S}+C\overset{\prime\prime\prime}{S}+D\overset{\text{iv}}{S}\quad (q+1);$$

$$\text{A being} = \frac{c-r.\ \ d-r.\ e-r\ (p)}{s-r.\ \ t-r.\ v-r\ (q)},\ \text{B}=\frac{c-s.\ d-s.\ e-s\ (p)}{r-s.\ t-s.\ v-s\ (q)},$$

$$\text{C} = \frac{c-t.\ \ d-t.\ e-t\ (p)}{r-t.\ \ s-t.\ v-t\ (q)},\ (q+1);$$

p not greater than q;

r, s, t, v, &c. any unequal numbers;

$$\overset{\prime}{S}=\frac{\overset{\prime}{P}x^{m}}{r}+\frac{P\,x^{m+n}}{r+n}+\frac{\overset{\prime\prime\prime}{P}x^{m+2n}}{r+2n}+\frac{\overset{\text{iv}}{P}x^{m+3n}}{r+3n}\ \&c.$$

$$\overset{\prime\prime}{S}=\frac{\overset{\prime}{P}x^{m}}{s}+\frac{\overset{\prime\prime}{P}x^{m+n}}{s+n}+\frac{\overset{\prime\prime\prime}{P}x^{m+2n}}{s+2n}+\frac{\overset{\text{iv}}{P}x^{m+3n}}{s+3n}\ \&c.$$

$$\overset{\prime\prime\prime}{S}=\frac{\overset{\prime}{P}x^{m}}{t}+\frac{\overset{\prime\prime}{P}x^{m+n}}{t+n}+\frac{\overset{\prime\prime\prime}{P}x^{m+2n}}{t+2n}+\frac{\overset{\text{iv}}{P}x^{m+3n}}{t+3n}\ \&c.$$

$$(q+1)\qquad\qquad (q+1).$$

But it is obvious, (as obſerved by Mr. Simpson, at page 147 of his *Diſſertations*,) that when x is $= a$,

$$\text{the ſeries } \overset{\prime}{S} \text{ is} = a^{m-r}\times fl.\,y\,x^{r-m-1}\dot{x},$$

$$\overset{\prime\prime}{S} = a^{m-s}\times fl.\,y\,x^{s-m-1}\dot{x},$$

$$\overset{\prime\prime\prime}{S} = a^{m-t}\times fl.\,y\,x^{t-m-1}\dot{x},$$

$$(q+1)\qquad\qquad (q+1);$$

y being $= \overset{\prime}{\mathrm{P}} x^{m} + \overset{\prime\prime}{\mathrm{P}} x^{m+n} + \overset{\prime\prime\prime}{\mathrm{P}} x^{m+2n} + \overset{\mathrm{iv}}{\mathrm{P}} x^{m+3n}$ &c.
and the fluents being generated whilſt x from o becomes $= a$.

Therefore, when x is $= a$,

the ſeries Q will be $= a^{m} \times fl.\ \frac{y\dot{x}}{x^{m+1}} \times \overline{\frac{\mathrm{A}x^{r}}{a^{r}} + \frac{\mathrm{B}x^{s}}{a^{s}} + \frac{\mathrm{C}x^{t}}{a^{t}}}\,(q+1)$,

generated in the time juſt now mentioned.

The illuſtration of this general theorem, and particularly the deducing corollaries therefrom, exhibiting comparative concluſions, with reſpect to Mr. LORGNA's propoſitions, will be the principal buſineſs of the following pages.

Note. Mr. SIMPSON has remarked, that it may ſometimes be of uſe to recollect, that

$$\mathrm{Q} = a^{m} \times fl.\ \frac{y\dot{x}}{x^{m+1}} \times \overline{\frac{\mathrm{A}x^{r}}{a^{r}} + \frac{\mathrm{B}x^{s}}{a^{s}} + \frac{\mathrm{C}x^{t}}{a^{t}}}\ (q+1)$$

$$\text{is} = \frac{a^{m}y}{x^{m}} \times \overline{\frac{\mathrm{A}x^{r}}{r-m.a^{r}} + \frac{\mathrm{B}x^{s}}{s-m.a^{s}}}\,(q+1) - a^{m} \times fl.\frac{\dot{y}}{x^{m}} \times \overline{\frac{\mathrm{A}x^{r}}{r-m.a^{r}} + \frac{\mathrm{B}x^{s}}{s-m.a^{s}}}\,(q+1).$$

II.

If y be taken $= \frac{1}{1 \mp x} = 1 \pm x + x^{2} \pm x^{3}$ &c.

m will be $= o$, $n = 1$, $\overset{\prime}{\mathrm{P}} = 1$, $\overset{\prime\prime}{\mathrm{P}} = \pm 1$, $\overset{\prime\prime\prime}{\mathrm{P}} = 1$, $\overset{\mathrm{IV}}{\mathrm{P}} = \pm 1$, &c.
and we ſhall have

$$\mathrm{Q} = \frac{c\,d\,e\,(q)}{rst(q+1)} \pm \frac{c+1.\ d+1.\ e+1\ (p) \times a}{r+1.\ s+1.\ t+1\ (q+1)} + \frac{c+2.\ d+2.\ e+2\ (p) \times a^{2}}{r+2.\ s+2.\ t+2\,(q+1)}$$

$$\pm \frac{c+3.\ d+3.\ e+3\ (p) \times a^{3}}{r+3.\ s+3.\ t+3\ (q+1)} \quad \text{\&c.}$$

$$= fl.\ \frac{\dot{x}}{1 \mp x} \times \overline{\frac{\mathrm{A}x^{r-1}}{a^{r}} + \frac{\mathrm{B}x^{s-1}}{a^{s}} + \frac{\mathrm{C}x^{t-1}}{a^{t}}}\,(q+1),$$

generated whilſt x from o becomes $= a$.

A,

A, B, C, &c. p, q, r, s, t, &c. being as in the preceding article.

Before I proceed to deduce the intended corollaries, it is neceſſary to obſerve, that, when $s-r$, $t-r$, $v-r$, &c. are poſitive integers,

$$\frac{x^{s-1}}{1-x} \text{ is } = -x^{r-1} - x^{r} - x^{r+1} - x^{r+2}\ (s-r) + \frac{x^{r-1}}{1-x},$$

$$\frac{x^{t-1}}{1-x} = -x^{r-1} - x^{r} - x^{r+1} - x^{r+2}\ (t-r) + \frac{x^{r-1}}{1-x},$$

$$(q+1) \qquad\qquad\qquad\qquad (q+1);$$

$$\text{and } Q = \overline{A a^{-r} + B a^{-s} + C a^{-t} + D a^{-v}\ (q+1)} \times G'$$

$$- B a^{r-s} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}\ (s-r)}$$

$$- C a^{r-t} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}\ (t-r)}$$

$$- D a^{r-v} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}\ (v-r)}$$

&c.

the upper ſign taking place (where there are double ones) in the ſeries Q;

and $\acute{G}$ being put to denote the fluent of $\frac{x^{r-1}\dot{x}}{1-x}$,

generated whilſt x from o becomes $= a$.

Therefore, if a be a root of the equation,

$$A a^{-r} + B a^{-s} + C a^{-t} + D a^{-v}\ (q+1) = o,$$

and $s-r$, $t-r$, $v-r$, &c. poſitive integers,

the value of the ſeries Q will be aſſigned in algebraic terms.

(See the final remark.)

It follows from what Mr. Simpson has obſerved at page 77 of his *Diſſertations*, that, if p be leſs than q, $A + B + C + D\ (q+1)$ will be $= o$; and that 1 will then be a value of a in the equation

A a

$\mathrm{A}a^{-r}+\mathrm{B}a^{-s}+\mathrm{C}a^{-t}(q+1)=0$. Conſequently, if $s-r, t-r, v-r$, &c. be *even* poſitive numbers, -1 will alſo be a value of a in that equation. Beſides which values, a may (in that equation) have others, that may be determined when the particular values of $s-r, t-r, v-r$, &c. are given.

It may be obſerved, that our theorum, (which in general comprehends a great deal more,) now y is taken of the particular value $\frac{1}{1\mp x}$, compriſes all the ſeries in Mr. LORGNA's work, except thoſe in his IXth ſection; (of which I ſhall take ſome notice by and by;) but, to make the uſe of the theorem (when y has that value) more clearly underſtood, I ſhall adapt it to the purpoſe of moſt of the ſections of that gentleman's book; and ſubmit it to the mathematical world to determine, upon comparing theſe Obſervations with that author's Diſſertation, whether either he, or his commentator, has any great reaſon to boaſt of ſuperior elegance and ſimplicity.

COROLLARY I. If p be $=0$, and $q=1$, A will be $=\frac{1}{s-r}$, $\mathrm{B}=-\frac{1}{s-r}$; C, D, E, &c. each $=0$: and our theorem, adapted to the purpoſe of Mr. LORGNA's ſecond ſection, will be

$$Q=\frac{1}{rs}\pm\frac{a}{r+1.\ s+1}+\frac{a^2}{r+2.\ s+2}\pm\frac{a^3}{r+3.\ s+3}\ \&c.$$

$$=\frac{1}{s-r}\times fl.\frac{\dot{x}}{1\mp x}\times\overline{\frac{x^{r-1}}{a^r}-\frac{x^{s-1}}{a^s}},$$

generated whilſt x from 0 becomes $=a$.

$$\text{Now } fl.\frac{\dot{x}}{1-x}\times\overline{\frac{x^{r-1}}{a^r}-\frac{x^{s-1}}{a^s}}\ \text{being}=\left\{\begin{array}{l}\overline{a^{-r}-a^{-s}}\times \mathrm{G}'\\ +a^{r-s}\times\overline{\frac{1}{r}+\frac{a}{r+1}(s-r)},\end{array}\right.$$

when $s-r$ is a poſitive integer;

C   and

and a, in the equation $a^{-r} - a^{-s} = 0$, or $a^{s-r} - 1 = 0$, being $= 1$; the ſeries Q, when a has that value and $s-r$ is any ſuch integer, will be $= \frac{1}{s-r} \times \overline{\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2}\ (s-r)}$:

alſo, -1 being a value of a in the ſame equation $(a^{s-r} - 1 = 0)$, when $s-r$ is an *even* poſitive number; the ſeries Q, when a is therein taken $= -1$, and $s-r$ is any ſuch number, will be $= \frac{1}{s-r} \times \overline{\frac{1}{r} - \frac{1}{r+1} + \frac{1}{r+2} - \frac{1}{r+3}\ (s-r)}$.

In other caſes, the value of Q will be aſſigned by circular arcs and logarithms.

Example I. With reſpect to the ſeries

$$\frac{1}{r.\,\overline{r+3}} + \frac{1}{\overline{r+1}.\,\overline{r+4}} + \frac{1}{\overline{r+2}.\,\overline{r+5}} + \frac{1}{\overline{r+3}.\,\overline{r+6}}\ \&c. = Q;$$

$s-r$ is $= 3$, and $Q = \frac{1}{3} \times \overline{\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2}}$.

Example **II.** If the ſeries whoſe ſum is ſought be

$$\frac{1}{r.\,\overline{r+2}} \pm \frac{1}{\overline{r+1}.\,\overline{r+3}} + \frac{1}{\overline{r+2}.\,\overline{r+4}} \pm \frac{1}{\overline{r+3}.\,\overline{r+5}}\ \&c. = Q;$$

$s-r$ will be $= 2$, and $Q = \frac{1}{2} \times \overline{\frac{1}{r} \pm \frac{1}{r+1}}$.

Example **III.** The ſeries $\frac{1}{r.\,\overline{r+1}} + \frac{1}{\overline{r+1}.\,\overline{r+2}} + \frac{1}{\overline{r+2}.\,\overline{r+3}}$ &c. $(= Q)$ being propoſed to be ſummed;

we have $s-r = 1$, and $Q = \frac{1}{r}$.

If r be $= \frac{2}{5}$, $Q = \frac{5\cdot 5}{2\cdot 7} + \frac{5\cdot 5}{7\cdot 12} + \frac{5\cdot 5}{12\cdot 17}$ &c. will be $= \frac{5}{2}$, and conſequently $\frac{1}{2.7} + \frac{1}{7.12} + \frac{1}{12.17}$ &c. $= \frac{1}{2.5}$:

Example IV. To find (Q) the ſum of the ſeries

$$\frac{1}{1.2} + \frac{1}{4.5} + \frac{1}{7.8} + \frac{1}{10.11} \text{ \&c.}$$

r muſt be taken $= \frac{1}{3}$, $s = \frac{2}{3}$; and we ſhall have

$$Q = \tfrac{1}{3} \times fl.\ \frac{x^{-\frac{2}{3}}\dot{x} - x^{-\frac{1}{3}}\dot{x}}{1-x} = \tfrac{1}{3} \times fl.\ \overline{1 - x^{\frac{1}{3}}} \times \frac{x^{-\frac{2}{3}}\dot{x}}{1-x},$$

generated whilſt x from o becomes $= 1$:

which fluent, by Theor. XV. of my Vth TABLE, is $= \frac{S}{t}$;
S denoting the ſemi-circumference of the circle whoſe rad. is 1, and t the tangent of $\frac{S}{3}$.

Example V. The ſum (Q) of the ſeries $\frac{1}{3.5} + \frac{1}{5.8} + \frac{1}{7.11}$ &c. being required;

we have $r = \frac{3}{2}$, $s = \frac{5}{3}$; and $Q = fl.\ \frac{x^{\frac{1}{2}}\dot{x} - x^{\frac{2}{3}}\dot{x}}{1-x}$,

generated whilſt x from o becomes $= 1$.

Now, if we ſubſtitute y^6 inſtead of x, it will appear that

$$Q \text{ is} = 6 \times fl.\ \frac{y^8\dot{y} - y^9\dot{y}}{1-y^6} = 6 \times fl.\ \overline{y^3\dot{y} - y^2\dot{y} + \frac{y^2\dot{y} - y^3\dot{y}}{1-y^6}}$$

$$= \frac{3y^4}{2} - 2y^3 + \text{Log.}\ \frac{1+y^3}{1-y^3} + \text{Log.}\ \overline{1-y^6} - \tfrac{3}{2}\ \text{Log.}\ \overline{1+y^2+y^4}$$

$$+ \sqrt{3} \times \text{circ. arc, rad, } 1\text{, tang. } \frac{3^{\frac{1}{2}}y^2}{2+y^2}$$

=

$$={}^{*}\frac{3y^{4}}{2}-2y^{3}+2\,\text{Log.}\,\overline{1+y^{3}}-\tfrac{1}{2}\,\text{Log.}\,\overline{1+y^{2}+y^{4}}$$

$$+\sqrt{3}\times\text{circ. arc, rad. }1\text{, tang. }\frac{3^{\frac{1}{2}}y^{2}}{2+y^{2}},$$

when y is therein taken $=1$.

Conſequently, by taking y of that value, we find the ſeries $Q=-\frac{1}{2}+2\,\text{Log.}\,2-\frac{3}{2}\,\text{Log.}\,3$

$$+\sqrt{3}\times\text{circ. arc, rad. }1\text{, tang. }\frac{1}{\sqrt{3}}.$$

Corollary II. If p be $=0$ and $q=2$, A will be $=\frac{1}{s-r.\,t-r}$, B $=\frac{1}{r-s.\,t-s}$, C $=\frac{1}{r-t.\,s-t}$; D, E, &c. each $=0$: and our theorem, adapted to the purpoſe of Mr. Lorgna's third ſection, will be

$$Q=\frac{1}{r\,s\,t}\pm\frac{a}{r+1.\,s+1.\,t+1}+\frac{a^{2}}{r+2.\,s+2.\,t+2}\,\&\text{c.}$$

$$=\text{fl.}\,\frac{\dot{x}}{1\mp x}\times\overline{\frac{a^{-r}x^{r-1}}{s-r.\,t-r}+\frac{a^{-s}x^{s-1}}{r-s.\,t-s}+\frac{a^{-t}x^{t-1}}{r-t.\,s-t}},$$

generated whilſt x from 0 becomes $=a$:

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* Mr. Lorgna erroneouſly makes the value of Q near four times as great as it really is: — his error ariſes from his conſidering the Log. of $\frac{1-x^{\frac{1}{3}}}{1-x^{\frac{1}{2}}}$ as $=0$, when x is $=1$; whereas that expreſſion is then $=$ Log. $\frac{2}{3}$, $\frac{1-x^{\frac{1}{3}}}{1-x^{\frac{1}{2}}}$ being always $=\frac{1+x^{\frac{1}{6}}}{1+x^{\frac{1}{6}}+x^{\frac{1}{3}}}$.

In the following part of his work he has made ſome other ſuch miſtakes; which will be pointed out in the courſe of theſe Obſervations, his commentator having not corrected any one of them!

which expreſſion,* when $s-r$, $t-r$ are poſitive integers, and a is a root of the equation,

$$\frac{a^{-r}}{\overline{s-r}.\overline{t-r}}+\frac{a^{-s}}{\overline{r-s}.\overline{t-s}}+\frac{a^{-t}}{\overline{r-t}.\overline{s-t}}=0,$$

will, by what is obſerved above, be

$$=\frac{1}{\overline{s-r}.\overline{t-s}}\times\overline{\frac{1}{r}+\frac{a}{r+1}+\frac{a}{r+2}}\,(s-r)-\frac{1}{\overline{r-t}.\overline{s-t}}\times\overline{\frac{1}{r}+\frac{a}{r+1}+\frac{a^2}{r+2}}\,(t-r).$$

When $s-r$, $t-r$ are any poſitive integers, a, in the above equation, will always be $=1$; and, when $s-r$, $t-r$ are any *even* poſitive numbers, a will alſo be $=-1$: beſides which values, a may (in the ſame equation) have others, that may be determined when the particular values of $s-r$, $t-r$ are given; for inſtance, when $s-r$ is $=1$ and $t-r=3$, a (in the ſaid equation) will be $=-\frac{1}{2}$.

Example I. If the ſeries

$$(Q=)\ \frac{1}{r.\overline{r+1}.\overline{r+3}}+\frac{a}{\overline{r+1}.\overline{r+2}.\overline{r+4}}+\frac{a^2}{\overline{r+2}.\overline{r+3}.\overline{r+5}}\ \&c.$$

be propoſed to be ſummed when a is $=1$, or $a=-\frac{1}{2}$:

$s-r$ being $=1$ and $t-r=3$,

we have, in the former caſe,

$$Q=\frac{1}{2}\times\frac{1}{r}-\frac{1}{6}\times\overline{\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}}=\frac{1}{6}\times\frac{3r+4}{r.\overline{r+1}.\overline{r+2}};$$

and, in the latter caſe, $Q=\frac{2}{3}\times\frac{1}{r.\overline{r+1}.\overline{r+2}}$.

Example II. Let the ſeries propoſed be

$$\frac{1}{2.3.7}+\frac{1}{3.4.10}+\frac{1}{4.5.13}\ \&c.=Q.$$

Then will a be $=1$, $r=2$, $s=3$, $t=\frac{7}{3}$;

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* Suppoſing the upper ſign to take place in the denominator $1\mp x$.

$$\text{and } Q = \text{fl.}\ \frac{x\dot{x}+\frac{1}{2}x^{2}\dot{x}-\frac{1}{2}x^{\frac{4}{3}}\dot{x}}{1-x}$$

$$= \tfrac{3}{2}\times\text{fl.}\ \frac{\dot{x}-x^{-\frac{2}{3}}\dot{x}}{1-x}-\dot{x}-\frac{x\dot{x}}{3}+x^{\frac{1}{3}}\dot{x}+x^{-\frac{2}{3}}\dot{x},$$

the fluents being generated whilſt x from o becomes $= 1$.

Now, if we ſubſtitute y^3 inſtead of x, we ſhall have

$$Q = \tfrac{9}{2}\times\text{fl.}\ \frac{y^2\dot{y}-\dot{y}}{1-y^3}-y^2\dot{y}-\frac{y^5\dot{y}}{3}+y^3\dot{y}+\dot{y}$$

$$= {}^{*}\frac{9y}{2}+\frac{9y^4}{8}-\frac{y^6}{4}-\frac{3y^3}{2}-\tfrac{9}{4}\text{ Log. }\overline{1+y+y^2}$$

$$-\frac{3^{\frac{3}{2}}}{2}\times\text{ circ. arc, rad. } 1,\text{ tang.}\frac{3^{\frac{1}{2}}y}{2+y},$$

when y is therein taken $= 1$.

Conſequently, by taking y of that value, we find

$$Q = \frac{31}{8}-\frac{9}{4}\text{Log. } 3-\frac{3^{\frac{3}{2}}}{2}\times\text{ circ. arc, rad. } 1,\text{ tang.}\frac{1}{\sqrt{3}}.$$

COROLLARY III. If p be $= o$, and $q = 3$, A will be $= \frac{1}{s-r.t-r.v-r}$, B $= \frac{1}{r-s.t-s.v-s}$, C $= \frac{1}{r-t.s-t.v-t}$, D $= \frac{1}{r-v.s-v.t-v}$; E, F, &c. each $= o$: and our theorem, adapted to the purpoſe of

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* Mr. LORGNA, in ſumming the ſeries (*Ex.* VIII, *pa.* 47) whoſe value is $\frac{1}{8}$ of this value of Q, makes a miſtake, by conſidering the Log. of $\frac{1-y}{1-y^3}$ as $= o$, when y is $= 1$; for that expreſſion will then be $=$ Log. $\frac{1}{3}$, $\frac{1-y}{1-y^3}$ being always $= \frac{1}{1+y+y^2}$.

In conſequence of which miſtake, he makes the ſum of his ſeries about *forty* times as great as it really is!

Mr. Lorgna's fourth ſection, will be

$$Q = \frac{1}{r\,s\,t\,v} \pm \frac{a}{r+1.\,s+1.\,t+1.\,v+1} + \frac{a^2}{r+2.\,s+2.\,t+2.\,v+2}, \text{ \&c.}$$

$$= fl.\, \frac{\dot{x}}{1 \mp x} \times \overline{\frac{a^{-r}x^{r-1}}{s-r.\,t-r.\,v-r} + \frac{a^{-s}x^{s-1}}{r-s.\,t-s.\,v-s} + \frac{a^{-t}x^{t-1}}{r-t.\,s-t.\,v-t} + \frac{a^{-v}x^{v-1}}{r-v.\,s-v.\,t-v}},$$

generated whilſt x from o becomes $= a$:

which expreſſion*, when $s-r$, $t-r$, $v-r$ are poſitive integers, and a is a root of the equation

$$\frac{a^{-r}}{s-r.t-r.v-r} + \frac{a^{-s}}{r-s.t-s.v-s} + \frac{a^{-t}}{r-t.s-t.v-t} + \frac{a^{-v}}{r-v.s-v.t-v} = o,$$

will, by what is obſerved above, be

$$= -\frac{a^{r-s}}{r-s.\,t-s.\,v-s} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}}\,(s-r)$$

$$-\frac{a^{r-t}}{r-t.\,s-t.\,v-t} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}}\,(t-r)$$

$$-\frac{a^{r-v}}{r-v.\,s-v.\,t-v} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}}\,(v-r).$$

When $s-r$, $t-r$, $v-r$ are any poſitive integers, a, in the above equation, will always be $= 1$; and, when $s-r$, $t-r$, $v-r$ are any *even* poſitive numbers, a will alſo be $= -1$: beſides which values, a may (in the ſame equation) have others, that may be determined when the particular values of $s-r$, $t-r$, $v-r$ are given; for inſtance, when $s-r$ is $= 1$, $t-r = 2$, $v-r = 4$, a (in the ſaid equation) will be $= -\frac{1}{3}$; and for another inſtance, when $s-r$ is $= 1$, $t-r = 3$, $v-r = 4$, a will be $= -1$.

(See the final remark.)

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* Suppoſing the upper ſign to take place in the denominator $1 \mp x$.

Example I. The ſeries

$$(Q=)\frac{1}{r.r+1.r+2.r+4}+\frac{a}{r+1.r+2.r+3.r+5}+\frac{a^2}{r+2.r+3.r+4.r+6}$$
&c.

being propoſed to be ſummed when a is $=1$, or $a=-\frac{1}{3}$:

$s-r$ being $=1$, $t-r=2$, $v-r=4$,

we have, in the former caſe,

$$Q=\frac{1}{3}\times\frac{1}{r}-\frac{1}{4}\times\overline{\frac{1}{r}+\frac{1}{r+1}}+\frac{1}{24}\times\overline{\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}}$$

$$=\frac{1}{12}\times\frac{4r+9}{r.r+1.r+2.r+3};$$

and, in the latter caſe,

$$Q=-\frac{1}{r}-\frac{9}{4}\times\overline{\frac{1}{r}-\frac{1}{3.r+1}}+\frac{27}{8}\times\overline{\frac{1}{r}-\frac{1}{3.r+1}+\frac{1}{9.r+2}-\frac{1}{27\,r+3}}$$

$$=\frac{3}{4}\times\frac{1}{r.r+1.r+2.r+3}.$$

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Example II. The ſeries propoſed being

$$\frac{1}{r.r+1.r+3.r+4}+\frac{a}{r+1.r+2.r+4.r+5}+\frac{a^2}{r+2.r+3.r+5.r+6}\text{\&c.}(=Q);$$

to find its ſum when a is $=1$, or $a=-1$: we have

$s-r=1$, $t-r=3$, $v-r=4$;

and accordingly

$$Q=\pm\frac{1}{6}\times\frac{1}{r}\mp\frac{1}{6}\times\overline{\frac{1}{r}\pm\frac{1}{r+1}+\frac{1}{r+2}}+\frac{1}{12}\times\overline{\frac{1}{r}\pm\frac{1}{r+1}+\frac{1}{r+2}\pm\frac{1}{r+3}}$$

$$=\frac{1}{6}\times\frac{2r+3}{r.r+1.r+2.r+3}\text{, or }=\frac{1}{2}\times\frac{1}{r.r+1.r+2.r+3}{}^{*},$$

according as a is $=1$, or $=-1$.

---

[*] Mr. LORGNA (at *pa.* 63) erroneouſly aſſerts, that, when a is $=-1$, the ſum cannot be expreſſed algebraically, becauſe $v-r$, $v-s$, $v-t$ are not *even* numbers: with what reaſon then, does he boaſt of having pointed out the *true criterion* of the poſſibility of the algebraic ſummation of a ſeries!

Example III. If it be required to find the ſum (Q) of the ſeries $\frac{1}{1.2.3.4}+\frac{1}{2.3.5.7}+\frac{1}{3.4.7.10}$ &c. we have $r=1$, $s=2$, $t=\frac{3}{2}$, $v=\frac{4}{3}$;

and $$Q = fl.\,\frac{\dot{x}-\frac{1}{2}x\dot{x}+4x^{\frac{1}{2}}\dot{x}-\frac{9}{2}x^{\frac{4}{3}}\dot{x}}{1-x}$$

$$=\frac{1}{2}\times fl.\,\overline{\frac{\dot{x}+8x^{-\frac{1}{2}}\dot{x}-9x^{-\frac{2}{3}}\dot{x}}{1-x}+\dot{x}-8x^{-\frac{1}{2}}\dot{x}+9x^{-\frac{2}{3}}\dot{x}},$$

the fluents being generated whilſt x from 0 becomes $=1$.

Now, if we ſubſtitute y^6 for x, we ſhall have

$$Q = 3\times fl.\,\overline{\frac{y^5\dot{y}+8y^2\dot{y}-9y\dot{y}}{1-y^6}+y^5\dot{y}-8y^2\dot{y}+9y\dot{y}}$$

$$=^{*}\frac{1}{2}y^6-8y^3+\frac{27y^2}{2}+8\,\text{Log.}\,\overline{1+y^3}-\frac{27}{4}\text{Log.}\,\overline{1+y^2+y^4}$$

$$-\frac{3^{\frac{5}{2}}}{2}\times \text{circ. arc, rad. 1, tang. }\frac{3^{\frac{1}{2}}y^2}{2+y^2},$$

when y is therein taken $=1$.

Conſequently, by taking y of that value, we find

$$Q = 6+8\,\text{Log.}\,2-\frac{27}{4}\,\text{Log.}\,3-\frac{3^{\frac{5}{2}}}{2}\times\text{circ. arc, rad. 1, tang. }\frac{1}{\sqrt{3}}.$$

---

* In giving this Example, Mr **Lorgna**, beſides the error which he has repeatedly committed reſpecting the value of Log. $\frac{1-y^g}{1-y^h}$ when y is $=1$, commits other errors in determining the coefficients of the fluxional expreſſions, and in aſſigning the fluents; which errors render his concluſion ſo abſurd as to give a negative quantity for the ſum of the ſeries whereof the terms are all poſitive! And his commentator, inſtead of correcting the miſtakes in the original, makes as many (or more) in his note!

Corollary IV. If p and q be each $= 1$, A will be $= \frac{c-r}{s-r}$, B $= -\frac{c-s}{s-r}$; C, D, &c. each $= 0$; and our theorem, adapted to the purpoſe of Mr. Lorgna's fifth ſection, will be

$$Q = \frac{c}{rs} \pm \frac{c+1 . a}{r+1 . s+1} + \frac{c+2 . a^2}{r+2 . s+2} \pm \frac{c+3 . a^3}{r+3 . s+3} \text{ \&c.}$$

$$= \frac{1}{s-r} \times \text{fl.}\ \frac{\dot{x}}{1 \mp x} \times \overline{\frac{c-r . x^{r-1}}{a^r} - \frac{c-s . x^{s-1}}{a^s}},$$

generated whilſt x from 0 becomes $= a$:

which expreſſion*, when $s-r$ is a poſitive integer, and a is a root of the equation, $\frac{c-r}{a^r} - \frac{c-s}{a^s} = 0$, or $a^{s-r} - \frac{c-s}{c-r} = 0$, will, by what is ſaid above, be

$$= \frac{c-s}{s-r} a^{r-s} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a}{r+2}}\ (s-r).$$

Example I. The ſeries (Q) propoſed to be ſummed being

$$\frac{r+2}{r . r+1} + \frac{r+4 . a}{r+1 . r+2} + \frac{r+6 . a^2}{r+2 . r+3} \text{ \&c.} = 2 \times \overline{\frac{\frac{1}{2}r+1}{r . r+1} + \frac{\frac{1}{2}r+2 . a}{r+1 . r+2} \text{ \&c.}}$$

we have $c = \frac{1}{2}r+1$, $s-r = 1$, $\frac{c-s}{c-r} = \frac{r}{r-2}$: and it appears, that, if a (leſs than 1) be $= \frac{r}{r-2}$, Q will be $= \frac{2-r}{r}$.

Example II. The ſeries (Q) whoſe ſum is required being

$$\frac{k-1}{k . k+2} + \frac{k a}{k+2 . k+4} + \frac{k+1 . a^2}{k+4 . k+6} \text{ \&c.} = \frac{1}{4} \times \overline{\frac{k-1}{\frac{1}{2}k . \frac{1}{2}k+1} + \frac{k . a}{\frac{1}{2}k+1 . \frac{1}{2}k+2} \text{\&c.}}$$

---

* Suppoſing the upper ſign to take place in the denominator $1 \mp x$.

c is $= k - 1$, $r = \frac{1}{2}k$, $s = \frac{1}{2}k + 1$, $s - r = 1$, $\frac{c-s}{c-r} = \frac{k-4}{k-2}$:

conſequently, if a (leſs than 1) be $= \frac{k-4}{k-2}$, Q will be $= \frac{k-2}{4k}$.

Example III. If the ſeries (Q) be

$$\frac{3r+8}{r.r+2} + \frac{3r+11.a}{r+1.r+3} + \frac{3r+14.a^2}{r+2.r+4} + \frac{3r+17.a^3}{r+3.r+5} \text{ \&c.}$$

$$= 3 \times \overline{\frac{r+\frac{8}{3}}{r.r+2} + \frac{r+\frac{11}{3}.a}{r+1.r+3} + \frac{r+\frac{14}{3}.a^2}{r+2.r+4} + \frac{r+\frac{17}{3}.a^3}{r+3.r+5} \text{ \&c.}}$$

c muſt be conſidered as $= r + \frac{8}{3}$, s as $= r + 2$: and it follows, that,

$$\text{if } a \text{ be } = \overline{\frac{c-s}{c-r}}\Big|^{\frac{1}{2}} = \pm\tfrac{1}{2},$$

$$\text{Q will be} = 4 \times \overline{\frac{1}{r} \pm \frac{1}{2.r+1}}.$$

Example IV. The ſeries propoſed being

$$\frac{1}{1.2} - \frac{3}{4.5} + \frac{5}{7.8} - \frac{7}{10.11} \text{ \&c.} = \text{Q};$$

we have $c = \frac{1}{2}$, $r = \frac{1}{3}$, $s = \frac{2}{3}$;

and $\text{Q} = \frac{1}{9} \times \textit{fl.}\ \overline{1+x}^{-\frac{1}{3}} \times \frac{x^{-\frac{1}{3}}\dot{x}}{1+x}$,

generated whilſt x from o becomes $= 1$:

which fluent, by Theor. VII. of my Vth. TABLE, is $= \frac{S}{s}$;

S denoting the ſemi-circumference of the circle whoſe rad. is 1,

and s the ſine of $\frac{2S}{3}$.

Corollary V. If p be $= 1$ and $q = 2$, A will be $= \frac{c-r}{s-r\,.\,t-r}$, B $= \frac{c-s}{r-s\,.\,t-s}$, C $= \frac{c-t}{r-t\,.\,s-t}$; D, E, &c. each $= 0$: and our theorem, adapted to the purpoſe of Mr. Lorgna's ſixth ſection, will be

$$Q = \frac{c}{r\,s\,t} \pm \frac{c+1\,.\,a}{r+1\,.\,s+1\,.\,t+1} + \frac{c+2\,.\,a^{2}}{r+2\,.\,s+2\,.\,t+2} \text{ \&c.}$$

$$= \text{fl. } \frac{\dot{x}}{1\mp x} \times \overline{\frac{c-r\,.\,a^{-r}\,x^{r-1}}{s-r\,.\,t-r} + \frac{c-s\;a^{-\ }\;x^{\ -1}}{r-s\,.\,t-s} + \frac{c-t\,.\,a^{-t}\,x^{t-1}}{r-t\,.\,s-t}},$$

generated whilſt x from 0 becomes $= a$:

which expreſſion*, when $s-r$, $t-r$ are poſitive integers, and a is a root of the equation

$$\frac{c-r\,.\,a^{-r}}{s-r\,.\,t-r} + \frac{c-s\,.\,a^{-s}}{r-s\,.\,t-s} + \frac{c-t\,.\,a^{-t}}{r-t\,.\,s-t} = 0,$$

will be $= -\frac{c-s\,.\,a^{r-s}}{r-s\,.\,t-s} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^{2}}{r+2} + \frac{a^{3}}{r+3}\,(s-r)}$

$$-\frac{c-t\,.\,a^{r-t}}{r-t\,.\,s-t} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^{2}}{r+2} + \frac{a^{3}}{r+3}\,(t-r)}.$$

When $s-r$, $t-r$ are any poſitive integers, a, in the above equation, will always be $= 1$; and when $s-r$, $t-r$ are any *even* poſitive numbers, a will alſo be $= -1$: beſides which values, a may (in the ſame equation) have others, that may be determined when the particular values of $s-r$, $t-r$ are given.

Example. The ſeries to be ſummed being

$$\frac{r-2}{r\,.\,r+2\,.\,r+4} + \frac{r-1\,.\,a}{r+1\,.\,r+3\,.\,r+5} + \frac{r\,a^{2}}{r+2\,.\,r+4\,.\,r+6} \text{ \&c.} = Q,$$

---

* Suppoſing the upper ſign to take place in the denominator $1\mp x$.

when a is $= 1$, or $a = -1$:

we have $c = r - 2$, $s = r + 2$, $t = r + 4$;

$$\text{and } Q = -\frac{1}{r} \mp \frac{1}{r+1} + \frac{3}{4} \times \overline{\frac{1}{r} \pm \frac{1}{r+1} + \frac{1}{r+2} \pm \frac{1}{r+3}}$$

$$= \frac{3}{4} \times \overline{\frac{1}{r+2} \pm \frac{1}{r+3}} - \frac{1}{4} \times \overline{\frac{1}{r} \pm \frac{1}{r+1}}$$

$$= \frac{1}{2} \times \frac{2r^3 + 5r^2 - r - 3}{r.r+1.r+2.r+3}, \text{ or} = \frac{1}{2} \times \frac{r^2 - r - 3}{r.r+1.r+2.r+3},$$

according as a is $= 1$, or $= -1$.

Corollary VI. If p be $= 2$ and $q = 3$, A will be $= \frac{c-r.d-r}{s-r.t-r.v-r}$, $B = \frac{c-s.d-s.}{r-s.t-s.v-s}$, $C = \frac{c-t.d-t}{r-t.s-t.v-t}$, $D = \frac{c-v.d-v}{r-v.s-v.t-v}$; E, F, &c. each $= 0$: and our theorem, adapted to the purpoſe of Mr. Lorgna's eighth ſection, will be

$$Q = \frac{cd}{rstv} \pm \frac{c+1.d+1.a}{r+1.s+1.t+1.v+1} + \frac{c+2.d+2.a^2}{r+2.s+2.t+2.v+2} \text{ \&c.}$$

$$= fl. \frac{\dot{x}}{1 \mp x} \times \overline{\frac{c-r.d-r.a^{-r}x^{r-1}}{s-r.t-r.v-r} + \frac{c-s.d-s.a^{-s}x^{s-1}}{r-s.t-s.v-s} + \frac{c-t.d-t.a^{-t}x^{t-1}}{r-t.s-t.v-t} + \frac{c-v.d-v.a^{-v}x^{v-1}}{r-v.s-v.t-v}},$$

generated whilſt x from 0 becomes $= a$:

which expreſſion*, when $s-r$, $t-r$, $v-r$ are poſitive integers, and a is a root of the equation

$$\frac{c-r.d-r.a^{-r}}{s-r.t-r.v-r} + \frac{c-s.d-s.a^{-s}}{r-s.t-s.v-s} + \frac{c-t.d-t.a^{-t}}{r-t.s-t.v-t} + \frac{c-v.d-v.a^{-v}}{r-v.s-v.t-v} = 0,$$

---

* Suppoſing the upper ſign to take place in the denominator $1 \mp x$.

will be $= -\frac{c-s.\,d-s.\,a^{r-s}}{r-s.\,t-s.\,v-s} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}\ (s-r)}$

$$-\frac{c-t.\,d-t.\,a^{r-t}}{r-t.\,s-t.\,v-t} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}\ (t-r)}$$

$$-\frac{c-v.\,d-v.\,a^{r-v}}{r-v.\,s-v.\,t-v} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} + \frac{a^3}{r+3}\ (v-r)}.$$

When $s-r$, $t-r$, $v-r$ are any poſitive integers, a, in the above equation, will always be $= 1$; and, when $s-r$, $t-r$, $v-r$ are any *even* poſitive numbers, a will alſo be $= -1$: beſides which values, a may (in the ſame equation) have others, that may be determined when the particular values of $s-r$, $t-r$, $v-r$ are given.

Example. Let the ſeries propoſed be

$$\frac{\overline{r-1}|^2}{r.\,r+1.\,r+2.\,r+3} + \frac{r^2}{r+1.\,r+2.\,r+3.\,r+4} + \frac{\overline{r+1}|^2}{r+2.\,r+3.\,r+4.\,r+5}\ \&c. = Q.$$

Then a being $= 1$, $c = d = r-1$, $s = r+1$, $t = r+2$, $v = r+3$; we have

$$Q = \frac{2^2}{2} \times \frac{1}{r} - \frac{3^2}{2} \times \overline{\frac{1}{r} + \frac{1}{r+1}} + \frac{4^2}{2.\,3} \times \overline{\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2}}$$

$$= \frac{1}{6} \times \frac{6r^2 - 3r + 2}{r.\,r+1.\,r+2}.$$

III.

With regard to Mr. LORGNA's ninth ſection, I have to obſerve, that the method of computation therein purſued may be derived from Mr. SIMPSON's propoſition at page 115 of his *Diſſertations*. Of which well-known method, neither Mr. BERNOULLI nor Mr. EULER can be ſuppoſed to have been ignorant; but we may ſuppoſe, that, with reſpect to the ſummation of the ſeries $1 \pm \frac{1}{2^k} + \frac{1}{3^k} \pm \frac{1}{4^k}$ &c. ($=Q$), when k is an odd number, that method was not found to ſupply the defects of their theorems according to their wiſhes, the obvious way of expreſſing the value of Q by the area of a tranſcendent hyperbola, and afterwards computing that area by the differential method*, being not at all ſatisfactory.

The *deſiderata* reſpecting the ſummation of ſuch ſeries, when k is an odd number, I conceive, are theorems like thoſe which were firſt given by Mr. John Bernoulli for finding the value of Q by means of *circular arcs*, when k is an even number; or theorems for finding the values of Q, in the former caſes, by means of the *arcs of any of the conic ſections.*

In the caſes wherein Mr. John Bernoulli's theorems fail, and indeed in all caſes whatever, the following theorem (derived from one at page 129 of my *Lucubrations*, and agreeing with that at page 678 of Mr. Maclaurin's *Fluxions*) enables us to compute the value of Q, to any degree of accuracy, with much leſs trouble than it can be computed by the tedious method injudiciouſly adopted by Mr. Lorgna, and unjuſtly extolled by his commentator: a method which even cannot be applied at all in computing the value of Q when k is a fraction; whereas the ſubſequent theorem is applicable let k be what it will.

As to the aſſigning the values of Q when Mr. John Bernoulli's theorems can be applied, they ſurely are, beyond all compariſon, preferable, at leaſt in point of elegance, to all others hitherto inveſtigated.

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* The ordinates being computed by repeated operations, by the ſame means as the areas, it is a troubleſome buſineſs to obtain, by that method, a concluſion ſufficiently exact.

THEOREM.

$$\frac{1}{\overline{a+d}|^{k}}+\frac{1}{\overline{a+3d}|^{k}}+\frac{1}{\overline{a+5d}|^{k}}+\frac{1}{\overline{a+7d}|^{k}}\text{ \&c. } (=Q) \text{ is}$$

$$=\frac{1}{2.\overline{k-1}.d\,a^{k-1}}-\frac{k\overset{\prime}{A}d}{2a^{k+1}}-\frac{\overline{2^{3}-1}\;k.k+1.k+2.\overset{\prime\prime}{A}d^{3}}{2.3.4\,a^{k+3}}$$

$$-\frac{\overline{2^{5}-1}.k.k+1.k+2.k+3.k+4.\overset{\prime\prime\prime}{A}d^{5}}{2.3.4.5.6\,a^{k+5}}\text{ \&c.}$$

$\overset{\prime}{A}$ being $=\frac{1}{2.3}=\frac{1}{6}$,

$\overset{\prime\prime}{A} = \frac{3}{2.5}-\frac{4}{2}\overset{\prime}{A}=-\frac{1}{30}$,

$\overset{\prime\prime\prime}{A} = \frac{5}{2.7}-\frac{6}{2}\overset{\prime}{A}-\frac{6.5.4}{2.3.4}\overset{\prime\prime}{A}=\frac{1}{42}$,

$\overset{iv}{A} = \frac{7}{2.9}-\frac{8}{2}\overset{\prime}{A}-\frac{8.7.6}{2.3.4}\overset{\prime\prime}{A}-\frac{8.7.6.5.4}{2.3.4.5.6}\overset{\prime\prime}{A}=-\frac{1}{30}$,

$\overset{v}{A} = \frac{9}{2.11}-\frac{10}{2}\overset{\prime}{A}-\frac{10.9.8}{2.3.4}\overset{\prime\prime}{A}-\frac{10.9.8.7.6}{2.3.4.5.6}\overset{\prime\prime\prime}{A}-\frac{10.9.8.7.6.5.4}{2.3.4.5.6.7.8}\overset{iv}{A}=\frac{5}{66}$,

&c. &c.

Example. To compute the ſum of the ſeries

$$*\,1+\frac{1}{2^{3}}+\frac{1}{3^{3}}+\frac{1}{4^{3}}\text{ \&c.}=Q:$$

find the ſum of the terms at the beginning of the ſeries as far as $\frac{1}{a+\frac{1}{2}}$ excluſively; then compute the ſubſequent terms by the above theorem, d being therein taken $=\frac{1}{2}$ and $k=3$. Thus, adding the firſt ten terms together, we find their ſum = 1.1975319856 &c. and, taking $a = \frac{21}{2}$, the ſum of the reſt of the terms is $= \frac{2}{\overline{21}|^{2}} \times \overline{1 - \frac{1}{\overline{21}|} + \frac{7}{3.\overline{21}|^{4}} - \frac{31}{3.\overline{21}|^{6}}}$ &c. Of which ſeries, only *four* terms (into the factor $\frac{2}{\overline{21}|}$) being taken, and their value (.0045249174 &c.) added to the ſum of the ten initial terms before found, we obtain 1.202056903 &c. for the value of Q, true to *nine* places of decimals. The ſame initial terms with *ſix* terms of Mr. De Moivre's ſeries give the ſame reſult: which is much more accurate than the reſult of Mr. Lorgna's computation of the ſum of the ſame ſeries, his value of Q being true only to *three* places of decimals. Yet, to magnify the comparative merit of his newly adopted method of computation, De Moivre's ſeries and others, into which our ſeries Q may be tranſmuted, are, by Mr. Clarke, decried as uſeleſs!

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* A new improvement, relative to the ſubject of this particular example, may be ſeen at the end of my Vth. Memoir.

IV.

It is well known that the ſum of the ſeries

$$Q = \frac{cde\,(p)}{rst\,(q+1)} + \frac{c+1.d+1.e+1\,(p) \times x}{r+1.s+1.t+1\,(q+1)} + \frac{c+2.d+2.e+2\,(p) \times x^{2}}{r+2.s+2.t+2\,(q+1)} \text{ \&c.}$$

will be infinite when x is $= 1$, if p be not leſs than q: of which Mr. Stirling was well aware, when he made the remark whereto Mr. Clarke refers in his *Preface*; which remark relates only to ſeries whoſe terms are all poſitive, as appears plainly by the examples afterwards given.

Mr. Simpson, in his Lemma referred to in Mr. Clarke's *Preface*, does not ſay, that the greateſt exponent of the indeterminate quantity in the numerator muſt be at leaſt two degrees lower than the greateſt exponent of the ſame quantity in the denominator, that the ſeries

may

may be ſummable:—when the ſum of a ſeries will be finite or infinite is not conſidered in that place; the writer's buſineſs, there, being to reſolve a compound fraction into as many ſimple ones as there are factors in its denominator; and, that his method may ſucceed, it is neceſſary (as he obſerves) that the number of factors in the numerator be *leſs* than the number of factors in the denominator, but not *leſs by two*, as Mr. Clarke ſeems to underſtand the paſſage.

When, in our ſeries Q, x is $= -1$, the value of ſuch ſeries, it is well-known, will be finite, even when p is $= q$.—I wonder therefore, that Mr. Lorgna and his commentator have, in ſo many inſtances, miſunderſtood the paſſages they allude to, reſpecting the impoſſibility of obtaining the ſums of certain ſeries; which, they would inſinuate, are ſummable only by means of their Formulæ! — Do they not, through their zeal for magnifying the importance of their favourite method, by ſuch inſinuation, *under-rate* the performances of others?—And do they not, through the ſame zeal, rather *over-rate* that method in ſaying, "that nothing "more can now be expected or wiſhed for in the Doctrine of "Series; being brought to its greateſt perfection"? — Perfection, in a doctrine ſo extenſive, ariſe from a method ſo narrow * and embarraſſed!— Surely their vaunts are vain!—And ſo indeed they would have been, even if their boaſted method had not been evidently leſs perfect than that which is exhibited above: — for though the theorem which I ſhall preſently add is much more comprehenſive than that in Art. II. which is ſhewn to comprehend the principal part of Mr. LORGNA's work; yet certainly much more is ſtill requiſite to bring the Doctrine of Series to its greateſt perfection, and render it as compleat as may be expected or wiſhed.

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* The ſeries ſummable by the method alluded to may, perhaps not improperly, be ſaid to be only of *one* claſs (in no reſpect untractable); the fluxions, whoſe fluents are in general requiſite in their ſummation, being all of the common form $\frac{x^g \dot{x}}{1 \mp x}$:—and when the ſummation in algebraic terms is poſſible, the ſame fluent may be conceived to be drawn into a number of ſuch terms whoſe ſum is $= b$.—The embarraſſment of the method is evident, from its failing to ſuggeſt the *true criterion* of the algebraic ſummation in all poſſible caſes.

THEOREM.

$$\frac{cde(p)\times\overset{\prime}{P}}{rst(q+1)}+\frac{c+1.d+1.e+1(p)\times\overset{\prime\prime}{P}a}{r+1.s+1.t+1(q+1)}+\frac{c+2.d+2.c+2(p)\times\overset{\prime\prime\prime}{P}a^2}{r+2.s+2.t+2(q+1)}\text{\&c.}$$

$$(=Q)\text{ is }=fl.\,\frac{y\dot{x}}{x}\times\overline{\frac{Ax^r}{a^r}+\frac{Bx^s}{a^s}+\frac{Cx^t}{a^t}}(q+1),$$

generated whilſt x from o becomes $=a$,

$$=\left.\begin{array}{l} Aa^{-r} \\ +\dfrac{r.r+1.r+2\,(s-r)\times Ba^{-s}}{k+r+1.k+r+2.k+r+3\,(s-r)\times b^{s-r}} \\ +\dfrac{r.r+1.r+2\,(t-r)\times Ca^{-t}}{k+r+1.k+r+2.k+r+3\,(t-r)\times b^{t-r}} \\ \qquad (q+1) \end{array}\right\}\times\overset{\prime}{G}$$

$$-\frac{r.r+1.r+2\,(s-r)\times B.\overline{1-ba}|^{k+1}}{k+r+1.k+r+2.k+r+3(s-r)\times\overline{ba}|^{s-r}}\times\left.\overline{\begin{array}{l}\dfrac{1}{r}+\dfrac{k+r+1.ba}{r.r+1}\\ \qquad+\dfrac{k+r+1.k+r+2.\overline{ba}|^2}{r.r+1.r+2}(s-\ )\end{array}}\right\},$$

$$-\frac{r.r+1.r+2\,(t-r)\times C.\overline{1-ba}|^{k+1}}{k+r+1.k+r+2.k+r+3(t-r)\times\overline{ba}|^{-r}}\times\left.\overline{\begin{array}{l}\dfrac{1}{r}+\dfrac{k+r+1.ba}{r.r+1}\\ \qquad+\dfrac{k+r+1.k+r+2.\overline{b.}|^2}{r.r+1.r+2}(t-r)\end{array}}\right\}$$

A, B, C, &c. being as in Art. I. $y = \overline{1-bx}|^k$.

$P = 1$, $P = -kb$, $\overset{\prime\prime\prime}{P} = k.\frac{k-1}{2}b^2$, $\overset{\text{iv}}{P} = k.\frac{k-1}{2}.\frac{k-2}{3}b^3$, &c.

and $'G = fl.\, \overline{1-bx}|^k \times x^{r-1}\dot{x}$,

generated whilſt x from 0 becomes $= a$.

Whence it appears, that, when $s-r$, $t-r$, $v-r$, &c. are any poſitive integers, and a is a root of the equation

$$Aa^{-r} + \frac{r.r+1\ (s-r) \times Bb^{s-r}a^{-s}}{k+r+1.k+r+2\ (s-r)} + \frac{r.r+1\ (t-r) \times Cb^{t-r}a^{-t}}{k+r+1.k+r+2\ (t-r)}\ (q+1) = 0,$$

the value of the ſeries Q will be aſſigned in algebraic terms, though the value of G′ may not be aſſignable in ſuch terms.

(See the final remark.)

Example. If b be $= 1$, $k = r-1$, $p = 0$, $q = 1$, and $s = r+1$; A will be $= 1$, $B = -1$; C, D, &c. each $= 0$: and the laſt equation becomes $a^{-r} - \frac{a^{-r-1}}{2} = 0$; where a is $= \frac{1}{2}$.

It appears therefore, that, when a is $= \frac{1}{2}$, the ſum of the ſeries

$$\frac{1}{r.r+1} - \frac{r-1.a}{r+1.r+2} + \frac{r-1.r\overset{-2.}{\wedge}a^2}{2.r+2.r+3} - \frac{r-1.r\overset{-2}{\wedge}.r-3.a^3}{2.3.r+3.r+4}\ \&c. \text{ is } = \frac{1}{2^r r}.$$

The general theorem here given, is derived from that in Art. I. and the ſecond in my VIIth TABLE: by which means ſome other theorems may likewiſe be deduced.

COROLLARY I. If (p being $= 0$,) s be $= 2r$, $t = 3r$, $v = 4r$ (q);

A will be $= \frac{1}{1.2.3\,(q) \times r^q}$, $B = -qA$, $C = \frac{q.q-1}{2}A$,

$D = -\frac{q.q-1.q-2}{2.3}A$ $(q+1)$;

and the ſeries $Q = \frac{1}{1.2.3\,(q) \times a^{qr+r}r^{q}} \times fl.\,\overline{a^{r}-x^{r}}|^{q} \times y\,x^{r-1}\dot{x}$,

generated whilſt x from o becomes $= a$,

let y $(= \overset{\prime}{P} + \overset{\prime\prime}{P}x + \overset{\prime\prime\prime}{P}x^{2} + \overset{\text{IV}}{P}x^{3}$ &c.) be what it may: which agrees with Mr. SIMPSON's *ſecond corollary*, at page 148 of his book ſo often mentioned.

Example. Let y be $= \overline{a-x}|^{k}$, $r = 1$, $s = 2$, $t = 3$, &c. then $Q \left(= \frac{a^{-q-1}}{1.\,2.\,3\,(q)} \times \right.$ the *whole* fluent of $\left. \overline{a-x}|^{k+q} \times \dot{x} \right)$ will be $= \frac{a^{k}}{1.\,2.\,3\,(q) \times \overline{k+q+1}}$, let k and q be what they will.

COROLLARY II. If (p being $= o$) s be $= r+1$, $t = r+2$, $v = r+3$, (q); A will be $= \frac{1}{1.\,2.\,3\,(q)}$, $B = -q\,A$, $C = \frac{q.q-1}{2}\,A$, $D = -\frac{q.q-1.q-2}{2.3}\,A$, $(q+1)$; and the ſeries

$$Q = \frac{a^{-q-r}}{1.2\ 3\,(q)} \times fl.\,y\,x^{r-1}\dot{x} \times \overline{a-x}|^{q},$$

generated whilſt x from o becomes $= a$.

Example. Let y be $= \frac{1}{1-x}$: then $(Q =)$

$$\frac{1}{r.r+1.r+2\,(q+1)} + \frac{a}{r+1.r+2.r+3\,(q+1)} + \frac{a^{2}}{r+2.r+3.r+4\,(q+1)} \text{ \&c.}$$

$$\left(= \frac{a^{-q-r}}{1.\,2.\,3\,q)} \times fl.\,\frac{x^{r-1}\dot{x} \times \overline{a-x}|^{q}}{1-x}, \right.$$

generated whilſt x from o becomes $= a$)

will be $=\frac{qA}{a}\times\frac{1}{r}-\frac{q.\,q-1.\,A}{2a^2}\times\overline{\frac{1}{r}+\frac{a}{r+1}}$

$$+\frac{q.\,q-1.\,q-2.\,A}{2.\,3a^3}\times\overline{\frac{1}{r}+\frac{a}{r+1}+\frac{a^2}{r+2}}\;(q)$$

$$+a^{-q-r}.\,\overline{a-1}\rceil^{q}.\,AG$$

$$=\frac{a^{-1}}{r.\,r+1.\,r+2.\,(q)\times q}+\frac{a^{-1}}{r.\,r+1.\,r+2(q-1)\times q.\,q-1}\times\frac{a-1}{a}$$

$$+\frac{a^{-1}}{r.\,r+1.\,r+2\,(q-2)\times q.\,q-1,\,q-2}\times\overline{\frac{a-1}{a}}\Big\rceil^{2}\;(q)$$

$$+a^{-q-r}.\,\overline{a-1}\rceil^{q}.\,AG;$$

G being $=$ *fl.* $\frac{x^{r-1}\dot{x}}{1-x}$, generated whilſt x from o becomes $=a$.

FINAL REMARK.

When q is greater than 2, the algebraic ſummation of the ſeries Q may be poſſible in other caſes beſides thoſe pointed out above:— for ſuch ſummation will be effected, if $s-r$, $t-r$, $v-r$, &c. be not all poſitive integers; provided each of the quantities s, t, v, &c. exceeds ſome one or more of the quantities $r, s, t, v,$ &c. by ſome poſitive integer; and provided a value of a be the ſame in each equation ariſing upon putting each ſum of the co-efficients of each different fluent (not aſſignable in algebraic terms) $=o$; as many as poſſible of the fluents of the fluxions $\overline{1-bx}\rceil^{k}\times x^{s-1}\dot{x}$, $\overline{1-bx}\rceil^{k}\times x^{t-1}\dot{x}$, $\overline{1-bx}\rceil^{k}\times x^{v-1}\dot{x}$, &c. being found by compariſon with ſome one of thoſe fluents; and as many other as poſſible of the reſt of thoſe fluents being found by the like compariſon with ſome other of thoſe fluents, &c.

And this holds true, as well with reſpect to the theorem in Art. IV. b and k being of any values whatever; as to the theorem in Art. II. in which b is $=1$, and $k=-1$.

A ſingle inſtance, after what has been done, will, I preſume, ſufficiently explain my meaning.

The ſeries $\frac{1}{r\,s\,t\,v}+\frac{a}{r+1\,.\,s+1\,.\,t+1\,.\,v+1}+\frac{a^2}{r+2\,.\,s+2\,.\,t+2\,.\,v+2}$ &c. ($=Q$), is above found $= fl.\ \frac{\dot{x}}{1-x}\times\overline{\frac{Ax^{r-1}}{a^r}+\frac{Bx^{s-1}}{a^s}+\frac{Cx^{t-1}}{a^t}+\frac{Dx^{v-1}}{a^v}}$, generated whilſt x from 0 becomes $=a$: where A is $=\frac{1}{s-r\,.\,t-r\,.\,v-r}$, $B=\frac{1}{r-s\,.\,t-s\,.\,v-s}$, $C=\frac{1}{r-t\,.\,s-t\,.\,v-t}$, $D=\frac{1}{r-v\,.\,s-v\,.\,t-v}$.

But $fl.\ \frac{x^{s-1}\dot{x}}{1-x}$ is $=\overset{\prime}{G}-a^r\times\overline{\frac{1}{r}+\frac{a}{r+1}+\frac{a^2}{r+2}\,(s-r)}$,

and $fl.\ \frac{x^{v-1}\dot{x}}{1-x}=\overset{\prime\prime}{G}-a^t\times\overline{\frac{1}{t}+\frac{a}{t+1}+\frac{a^2}{t+2}\,(v-t)}$;

$\overset{\prime}{G}$ and $\overset{\prime\prime}{G}$ being the reſpective fluents of $\frac{x^{r-1}\dot{x}}{1-x}$ and $\frac{x^{t-1}\dot{x}}{1-x}$, generated in the time juſt now mentioned.

$$\text{Therefore } Q \text{ will be } = \overline{\frac{A}{a^r}+\frac{B}{a^s}}\times G+\overline{\frac{C}{a^t}+\frac{D}{a^v}}\times\overset{\prime\prime}{G}$$

$$-Ba^{r-s}\times\overline{\frac{1}{r}+\frac{a}{r+1}+\frac{a^2}{r+2}\,(s-r)}$$

$$-Da^{t-v}\times\overline{\frac{1}{t}+\frac{a}{t+1}+\frac{a^2}{t+2}\,(v-t)}:$$

conſequently, if $\frac{A}{a^r}+\frac{B}{a^s}$ be $=0$, and $\frac{C}{a^t}+\frac{D}{a^v}=0$; and only $s-r$, $v-t$ be poſitive integers; the value of the ſeries Q will be expreſſed in algebraic terms.

Now, that thoſe two may be true equations, a muſt be

$$=\overline{\frac{-B}{A}}\Big|^{\frac{1}{s-r}}=\overline{\frac{-D}{C}}\Big|^{\frac{1}{v-t}};\ \text{that is, } a=\overline{\frac{t-r\,.\,v-r}{t-s\,.\,v-s}}\Big|^{\frac{1}{s-r}}=\overline{\frac{r-t\,.\,s-t}{r-v\,.\,s-v}}\Big|^{\frac{1}{v-t}}.$$

Let r be $= b$, $s = b+1$, $t = b - \frac{1}{2}$, $v = b + \frac{3}{2}$; and it will appear by ſubſtitution, that the ſeries

$$\frac{1}{b.b+1.2b-1.2b+3} + \frac{1}{b+1.b+2.2b+1.2b+5} + \frac{1}{b+2.b+3.2b+3.2b+7} \text{\&c.}$$

$$(= Q) \text{ is } = \frac{1}{3} \times \overline{\frac{1}{2b-1} + \frac{1}{2b+1}} - \frac{1}{3} \times \frac{1}{b} = \frac{1}{3} \times \frac{1}{b.2b-1.2b+1}.$$

Here again Mr. LORGNA's *pretended indubitable mark*, or *true criterion*, fails!—For, were we to judge by *that*, we ſhould conclude, that the ſum of this ſeries cannot poſſibly be expreſſed algebraically; $v-r$, $v-s$, $v-t$ being not all poſitive integers!—And many other inſtances of its fallibility might be given.

Having made theſe obſervations, I cannot but think, that the commentator would have ſhewn not leſs judgment, if, inſtead of applauding ſo extravagantly the faulty work which he undertook to explain, he had corrected the errors in it, and ſupplied its defects. A work, whereof more than *thirty* pages are filled with embarraſſed inveſtigations of ill-expreſſed, particular Formulæ; without exhibiting any general theorem, like our *Simpſonian* one at page 3; though boaſts of perſpicuity, and indeed of every excellence, are not wanting! Surely Mr. CLARKE has made this country a very poor compliment, in ſuppoſing, that ſuch a work, with his frivolous and tedious notes thereon, could be acceptable to the Britiſh mathematicians!

THE END.

ERRATA.

Pa. 3, l. 20, *for* Cx^{-1}, *read* Cx^{t-1}. P. 5, l. 6, *for* theorum, *read* theorem. P. 24, l. 2, *read* $\overset{\text{IV}}{P} = -k.\frac{k-1}{2}.\frac{k-2}{3}b^3$; and l. 7, *for* b^{s-r}, b^{t-r}, *read* b^{r-s}, b^{r-t}.

AN

APPENDIX

TO

OBSERVATIONS

ON

CONVERGING SERIES.

APPENDIX.

Mr. Clarke having attempted to defend his *Tranſlation* of Mr. Lorgna's *Treatiſe on Series* againſt my animadverſions thereon, I think it expedient to exerciſe my pen again on that ſubject, to ſhew how futile the defender's remarks are; which, whatever he may pretend, are manifeſtly dictated by *pique* and *chagrin*, and expreſſed in a manner ſerving only to *debaſe argument to cavil.*

The purpoſe of my writing is chiefly to vindicate the reputation of a deceaſed friend, whoſe valuable work is unjuſtly depreciated, that it may ſerve, in compariſon, to ſet off one of much leſs value, publiſhed under a *ſpecious* Title, and with a *boaſtful* Preface; promiſing to exhibit a method entirely new and much more general than any other which had before appeared on the ſubject, whilſt the contents of the book are very far from anſwering the expectation ſuch title and preface are calculated to raiſe: and I truſt, that the expoſing ſuch diſingenuity, and beſtowing applauſe where due, will not be deemed diſcommendable by the public in a writer whoſe own works are not brought into the compariſon.

In my *Obſervations* on Mr. Clarke's *Tranſlation*, I have only adapted Mr. Simpson's Theorem to the purpoſe of moſt of the ſections in

Mr. Lorgna's book: ſubmitting it to the Mathematical world to determine, upon comparing thoſe *Obſervations* with the contents of his book, whether either he or his commentator, has any great reaſon to boaſt of ſuperior elegance and ſimplicity. But the commentator, not ſatisfied with ſuch appeal, urges me (by his *Supplement*) to ſtate the matter in diſpute more fully. I am ready to do ſo; and to abide by the conſequent, impartial deciſion of the public.

I do not ſay that Mr. Simpson's method is the ſame as Mr. Lorgna's; but I contend that it is more perſpicuous and elegant than his, and not leſs (if not more) general.

The whole buſineſs of Mr. Lorgna's work (excepting his 9th ſection) is to ſolve one ſingle problem, (not very remarkable for its difficulty,) and to give ſome examples by way of illuſtrating the ſolution: which problem is, to find the ſum of the ſeries.

$$\frac{a+b.c+e(b)}{p+q.m+n.r+s(k)} \pm \frac{a+2b.c+2e(b)\times x}{p+2q.m+2n.r+2s(k)} + \frac{a+3b.c+3e\,(b)\times x^2}{p+3q.m+3n.r+3s(k)}\ \&c.$$

$$=\frac{b\,e\,(b)}{q\,n\,s\,(k)} \times \overline{\frac{\frac{a}{b}+1.\frac{c}{e}+1\,(b)}{\frac{p}{q}+1.\frac{m}{n}+1.\frac{r}{s}+1\,(k)} \pm \frac{\frac{a}{b}+2.\frac{c}{e}+2\,(b)\times x}{\frac{p}{q}+2.\frac{m}{n}+2.\frac{r}{s}+2\,(k)}}\ \&c.$$

Now, though he has written a great deal on the ſubject, he has, after all, only conſidered *b* and *k* of ſome particular values; and, conſequently, has only ſolved ſome particular caſes of the problem, without giving a general ſolution!

The ſame problem, Mr. Simpson in his *Diſſertations* has ſolved generally in a few pages!

There is indeed a little difference in the manner of writing the ſeries; but that makes no difference as to the generality of the ſolution, as will plainly appear preſently.

Mr. Simpson's ſeries (pa. 78), taking therein $n = 1$ and writing x

for z is $\frac{a\,b\,(b)}{p\,q\,r\,(k)} \pm \frac{a+1.b+1\,(b) \times x}{p+1.q+1.r+1\,(k)} + \frac{a+2.b+2.\,(b) \times x^2}{p+2.q+2.r+2\,(k)}$ &c. which being multiplied by a certain factor (whoſe value is obvious,) after writing $\frac{a}{b}+1$, $\frac{c}{e}+1$, &c. inſtead of a, b, &c. and $\frac{p}{q}+1$, $\frac{m}{n}+1$, $\frac{r}{s}+1$, &c. inſtead of p, q, r, &c. reſpectively, the product will be the ſame as Mr. LORGNA's ſeries.

Therefore Mr. CLARKE's remark (pa 49. of his ſupplement) reſpecting the generality of a ſeries, from its having *two* characters when *one* would be ſufficient, is extremely futile!—And, whatever he may think, I believe it will be allowed by competent judges of the ſubject, that the making uſe of many characters unneceſſarily is perplexing and inelegant; and, inſtead of conducing to perſpicuity, often occaſions embarraſſment. Of the truth of this obſervation, Mr. LORGNA's proceſs is perhaps an inſtance: wherein

$\frac{1}{\frac{p}{q}+1}$, $\frac{1}{\frac{p}{q}+2}$, &c. $\left(\text{or } \frac{q}{p+q}, \frac{q}{p+2q}, \text{ \&c.}\right)$ and $\frac{1}{\frac{m}{n}+1}$, $\frac{1}{\frac{m}{n}+2}$, &c. $\left(\text{or } \frac{n}{m+n}, \frac{n}{m+2n}, \text{ \&c.}\right)$ are unneceſſarily made uſe of inſtead of $\frac{1}{p}$, $\frac{1}{p+1}$, &c. and $\frac{1}{q}$, $\frac{1}{q+1}$, &c. And, what is aſtoniſhing! the ſupplement writer is ignorant, or inconſiderate enough to aſſert (pa. 48. 49.) that theſe ſimple expreſſions, with only one character in each, are not ſo comprehenſive as thoſe other, with two characters in each! as if p, q, &c. would not expreſs as much as $\frac{p}{q}$, $\frac{m}{n}$, &c. or $\frac{p}{q}+1$, $\frac{m}{n}+1$, &c. Such an aſſertion ſurely would even diſgrace a novice!

Mr. CLARKE's other remark relating to the generality of Mr. LORGNA's method, when y is taken $= x^{\frac{p}{q}} \times \overline{1-b\lambda}|^{t}$ inſtead of

x,

$\frac{x^{\frac{p}{q}}}{1 \mp x}$, is equally futile !—For if he changes his ground by ſubſtitution, (pa. 47. of his ſupplement) why may not I likewiſe change my ground, and be allowed to make a ſimilar ſubſtitution in the Theorem at pa. 3 of my *Obſervations?* And were I to do ſo, the reſult of that Theorem would certainly be much more general than his at pa. 48. of his ſupplement.—Indeed I have made a ſubſtitution for *y*, at. pa. 23. of my *Obſervations*, ſo nearly ſimilar to the above, that, by only altering a character or two, his concluſion (and many others) may be directly obtained from what I have there done :—of which he, very probably, availed himſelf, by diſguiſing my Theorem and applying it (pa. 8.) to his own purpoſe ; for it will ſcarcely be believed, that he actually found his Formula by induction.

Mr. Simpson, in ſolving the problem before mentioned, does not conſider how the propoſed Series may ariſe by taking fluents and fluxions, as Mr. Lorgna has done ; but he reſolves the Series which is to be ſummed into as many other Series (each having a ſimple denominator and being ſummable by a known Theorem) as there are factors in each denominator ; and by ſuch means he finds (in his 4th propoſition)

$$\frac{a\,b\,(b)}{p\,q\,r\,(k)} \pm \frac{\overline{a+n}.\overline{b+n}\,\overline{(b)} \times z^{n}}{\overline{p+n}.\overline{q+n}.\overline{r+n}\,\overline{(k)}} + \frac{\overline{a+2n}.\overline{b+2n}\,\overline{(b)} \times z^{2n}}{\overline{p+2n}.\overline{q+2n}.\overline{r+2n}} \pm \text{\&c.}$$

$$= \mathrm{A} \times \overline{\frac{1}{p} \pm \frac{z^{n}}{p+n} + \frac{z^{2n}}{p+2n} \pm \frac{z^{3n}}{p+3n}\ \text{\&c.}}$$

$$+ \mathrm{B} \times \overline{\frac{1}{q} \pm \frac{z^{n}}{q+n} + \frac{z^{2n}}{q+2n} \pm \frac{z^{3n}}{q+3n}\ \text{\&c.}}$$

$$+ \mathrm{C} \times \overline{\frac{1}{r} \pm \frac{z^{n}}{r+n} + \frac{z^{2n}}{r+2n} \pm \frac{z^{3n}}{r+3n}\ \text{\&c.}}$$

$$\text{\&c.}\ (k)$$

=

$$=\frac{A}{z^p}\times fl.\frac{z^{p-1}\dot{z}}{1\mp z^n}+\frac{B}{z^q}\times fl.\frac{z^{q-1}\dot{z}}{1\mp z^n}+\frac{C}{z^r}\times fl.\frac{z^{r-1}\dot{z}}{1\mp z^n}\quad(k)$$

b being leſs than k

$$A=\frac{a-p.b-p\,(b)}{q-p.r-p\,(k-1)},\ B=\frac{a-q.b-q\,(b)}{p-q.r-q\,(k-1)},\ C=\frac{a-r.b-r\,(b)}{p-r.q-r\,(k-1)},$$

&c. (k).

Thus eaſily, in four or five pages, does Mr. **Simpson** obtain a general ſolution to the problem; about which Mr. **Lorgna** has written a *Treatiſe*, without exhibiting any thing equally comprehenſive!

But to adapt ſuch ſolution to the caſes wherein $\frac{p-m}{n}$, $\frac{q-p}{n}$, $\frac{r-p}{n}$, &c. are unequal whole numbers, Mr. **Simpson** proceeds farther, and, denoting the fluent of $\frac{z^{m-1}\dot{z}}{1-z^n}$ by S, finds his propoſed Series in that caſe, (conſidering all its ſigns as affirmative),

$$=\frac{A}{z^p}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\cdots\cdots\cdots\frac{z^{p-n}}{p-n}}\left(\frac{p-m}{n}\right)$$
$$+\frac{B}{z^q}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\cdots\cdots\cdots\frac{z^{q-n}}{q-n}}\left(\frac{q-m}{n}\right)$$
$$+\frac{C}{z^r}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\cdots\cdots\cdots\frac{z^{r-n}}{r-n}}\left(\frac{r-m}{n}\right)$$

&c. (k)

where, when n is equal to, or greater than p, (ſuppoſed the leaſt of the quantities p, q, r, &c.) m muſt be $=$p*.

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* Mr. **Clarke** in his note pa. 45. of his ſupplement ſays that when $\frac{p-m}{n}$, $\frac{q-m}{n}$, &c. happen to be fractional or negative numbers; that is, if $p-n$, $q-n$, &c. be o or negative, S is no longer $= fl.\frac{z^{m-1}\dot{z}}{1-z^n}$, but is of different values in the theorem, viz. $fl.\frac{z^{p-1}\dot{z}}{1-z^n}$, $fl.\frac{z^{q-1}\dot{z}}{1-z^n}$, &c. and he thinks this is a neceſſary remark, though Mr. **Simpson** has omitted it. But here Mr. **Clarke** is miſtaken; his remark is unne-

In other caſes, m may be taken at pleaſure; ſo that $\frac{p-m}{n}$, the leaſt of the quantities $\frac{p-m}{n}$, $\frac{q-m}{n}$, $\frac{r-m}{n}$ &c. be $= o$ or a poſitive integer; and the reſt of thoſe quantities all integers.

If m be taken $= p$, the expreſſion $\frac{A}{z^p} \times \overline{\frac{z^m}{m} + \frac{z^{m+n}}{m+n}\left(\frac{p-m}{n}\right)}$ will vaniſh, and the theorem become ſomewhat more conciſe.

The *criterion* for the algebraic ſummation is obvious. For if $\frac{A}{z^p} + \frac{B}{z^q} + \frac{C}{z^r}$ (k) be $= o$, the value of the ſum of the ſeries will, it is plain, be purely algebraical: which by the *Cor.* to Mr. SIMPSON's *Lem.* (pa. 76 of his *Diſſert.*) will always be the caſe when, $k-b$ being $= 2$ or a greater number, the differences $q-p$, $r-p$, $s-p$, &c. are any integers and z is $= 1$; alſo when thoſe differences are all even numbers and z is $= -1$.

Moreover it is evident that there are other caſes in which the value of the ſeries may be purely algebraical, as the expreſſion $\frac{A}{z^p} + \frac{B}{z^q} + \frac{C}{z^r}$ (k) may be $= o$ when z has other values (beſides 1 and $-$ 1), that may be determined when the particular values of, p, q, r, &c. are given.

Thus at once we diſcover, by Mr. SIMPSON's proceſs, more than Mr. LORGNA after many operations, has diſcovered reſpecting the *algebraic* ſummation of the ſeries propoſed!

A remark of the like import with the final one in my *Obſervations* may

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ceſſary. For when p (the leaſt of the quantities p, q, r, &c.) is $= m$ and $\frac{q-p}{n}$, $\frac{r-p}{n}$, &c. are whole numbers, as they are ſuppoſed to be in the caſe conſidered by Mr. SIMPSON; none of the quantities q, r, &c. can be leſs than n.

may be made reſpecting the value of S: and from thence it will be evident that the algebraic ſummation of the propoſed ſeries may be poſſible though $\frac{q-p}{n}$, $\frac{r-p}{n}$, &c. be not all whole numbers; provided each of the quantities $\frac{q}{n}$, $\frac{r}{n}$, &c. exceeds ſome one or more of the quantities $\frac{p}{n}$, $\frac{q}{n}$, $\frac{r}{n}$, &c. by ſome poſitive integer; and the number of factors in the denominator of each term of the ſeries be greater than 3.

It is manifeſt therefore, that by Mr. SIMPSON's 4th propoſition (which is a particular caſe of his 5th) all Mr. LORGNA's ſeries (except thoſe in his 9th ſect.) may be ſummed with great facility:—and to that propoſition I might have referred in my *Obſervations*, inſtead of referring to the 5th, had I not intended to give a more general theorem than that which is propoſed to be inveſtigated in that 4th propoſition.

Mr. CLARKE (pa. 43. of his ſuppl.) talks of my putting Mr. SIMPSON's theorems to the rack, in order to extort a ſecret from them which they do not contain: but it is evident that, without any torture, thoſe theorems impart ſuch a ſecret as Mr. CLARKE may bluſh to find divulged!

Having taken a view of Mr. SIMPSON's method of proceeding, and obſerved the reſult of it, in the ſolution of the problem under conſideration; let us now take ſome farther notice of what Mr. LORGNA has done reſpecting a ſolution of the ſame problem. His method is very different from Mr. SIMPSON's.

He obſerves, that

$$F = \frac{1}{q}\,\text{ſl}.\,\frac{x^{p:q}\dot{x}}{1\mp x} \text{ is} = \frac{x^{p:q+1}}{p+q} \pm \frac{x^{p:q+2}}{p+2q} + \frac{x^{p:q+3}}{p+3q} \pm \text{ \&c.}$$

$$G = \frac{1}{n}\,\mathit{fl.}\,F\,x^{m:n-p:q-1}\dot{x} = \frac{x^{m:n+1}}{p+q.m+n} \pm \frac{x^{m:n+2}}{p+2q.m+2n} + \&c.$$

$$H = \frac{1}{s}\,\mathit{fl.}\,G\,x^{r:s-m:n-1}\dot{x} = \frac{x^{r:s+1}}{p+q.m+n.r+s} \pm \frac{x^{r:s+2}}{p+2q.m+2n.r+2s} + \&c.$$

&c: &c. &c.

which are obvious and well known equations; and the conſequence of repeating the operation, by which they are obtained, is alſo obvious and well known.

He likewiſe obſerves, that

$$B = \frac{1}{\dot{x}}\text{ flux. of } F\,bx^{a:b-p:q}\text{ is } = \frac{a+b.x^{a:b}}{p+q} \pm \frac{a+2b.x^{a:b+1}}{p+2q} + \&c.$$

$$C = \frac{1}{n}\,\mathit{fl.}\,B\,x^{m:n-a:b}\,\dot{x} = \frac{a+b.x^{m:n+1}}{p+q.m+n} \pm \frac{a+2b.x_{m:n}{}^{+2}}{p+2q.m+2n} + \&c.$$

&c. &c. &c.

And moreover, that

$$\overset{\prime}{D} = \frac{1}{\dot{x}}\text{ flux. of } C\,e\,x^{c:e-m:n}\text{ is } = \frac{a+b.c+e\,x^{c:e}}{p+q.m+n} \pm \frac{a+2b\,c+2e.x^{c:e+1}}{p+2q.m+2n}\,\&c.$$

$$\overset{\prime\prime}{E} = \frac{1}{s}\,\mathit{fl.}\,\overset{\prime\prime}{D}\,x^{r:s-c:e}\dot{x} = \frac{a+b.c+e.x^{r:s+1}}{p+q.m+n.r+s} \pm \frac{a+2b.c+2e.x^{r:s+2}}{p+2q.m+2n.r+2s}\,\&c.$$

&c. &c. &c.

All which equations are equally obvious and well known.

He then proceeds to find the values of the expreſſions G, H, &c. C, D, &c. $\overset{\prime\prime}{E}$, $\overset{\prime\prime}{F}$, &c. by ſucceſſive operations, by the help of the well known equation $(vw - \mathit{fl.}\,v\dot{w} = \mathit{fl.}\,w\dot{v})$ $z\,\mathit{fl.}\,y\dot{x} - \mathit{fl.}\,yz\dot{x} = \mathit{fl.}\,\overline{\dot{z}\,\mathit{fl.}\,y\dot{x}}$, which he premiſes as a Lemma: the value of F being conſidered as known; which, it may be obſerved, is what Mr. Simpson conſiders as known.

Thus

$$G = \frac{1}{n}\,\mathit{fl.}\,x^{m:n-p:q-1}\dot{x}\text{ being } = \frac{1}{nq}\,\mathit{fl.}\,x^{m:n-p:q-1}\dot{x}\,\mathit{fl.}\,\overline{\frac{x^{p:q}\dot{x}}{1\mp x}},$$

he, by dividing the laſt expreſſion into two, by help of his Lemma, finds

$$G = \frac{x^{m:n-p:q}}{nq \times m:n-p:q} \, fl. \, \frac{x^{p:q}\dot{x}}{1 \mp x} - \frac{1}{nq \times m:n-p:q} \, fl. \, \frac{x^{m:n}\dot{x}}{1 \mp x}.$$

The value of G being ſo found, he, by means thereof, in like manner, finds the value of H, &c.

Alſo, by taking the fluxions, the value of B is found from the expreſſion $\frac{1}{\dot{x}}$ flux. of F $b\,x^{a:b-p:q}$ (which denotes its value); and the value of C is then found by the help of his Lemma.

Likewiſe, by taking the fluxions, the value of $\overset{\prime\prime}{D}$ is found from the expreſſion $\frac{1}{\dot{x}}$ flux. of C $e\,x^{c:c-m:n}$; and, by help of his Lemma, the the value of $\overset{\prime\prime}{E}$ is then found. &c.

In which ſucceſſive operations, ſo many troubleſome ſubſtitutions are requiſite, and the embarraſſment, ariſing therefrom, ſo irkſome, that we need not wonder, former writers have not purſued the method to any great length, nor that Mr. LORGNA has left the ſolution of the problem incomplete. Surely no analyſt of judgment would purſue that method, after having ſeen Mr. SIMPSON's.

The formulæ which Mr. LORGNA deduces as above are, from the perplexity of the proceſs, moſt of them ill expreſſed; and none of them properly adapted for finding the *complete criterion of algebraic ſummation*, which it is vainly pretended he has pointed out! Nor are they better adapted for finding, by induction, a general theorem like Mr. SIMPSON's: for after all Mr. LORGNA has done, though he has *made a book* of his labours, he has not diſcovered any ſuch theorem! nor does it appear (ſee pa. 112.) that he had any notion of a general thereom being attainable by induction.—Indeed if he had ſo diſcovered ſuch theorem, it would not, as Mr. CLARKE himſelf ac-

know-

knowledges, pa. 209. have been ſo ſatisfactory as Mr. SIMPSON's, which reſults from a pure algebraic inveſtigation.

Is it not then a *vain boaſt* to talk (ſee Mr. CLARKE's *Preface*) of having brought the doctrine of ſeries to its greateſt perfection?— Mr. CLARKE himſelf proves, that it is a *mere puff!*—witneſs the additions in his *Supplement*; for which, undoubtedly, he will be thought to be *ſomewhat indebted to my obſervations*.

It is obſervable, that, with reſpect to ſeries having but two factors in each denominator of their terms, Mr. LORGNA has given us the *algebraic ſummation* in certain caſes when x is leſs than 1: why he did not do ſo, where the denominators conſiſt of more than two factors, is perhaps pretty plain; the taſk, becoming more difficult, required a method of preceeding more general than his. The *Simpſonian* method enables us to find all the values of x, under which ſuch ſummation is poſſible.

The aukwardneſs of Mr. LORGNA's formulæ pa. 62.63. is ſuch, that, from thence he has erroneouſly inferred, that, if the ſigns in the ſeries there conſidered be alternately + and −, the ſummation cannot be exhibited algebraically, unleſs $\frac{t}{u}-\frac{p}{q}$, $\frac{t}{u}-\frac{r}{s}$, and $\frac{t}{u}-\frac{m}{n}$ be *even* numbers: which inference is proved to be falſe at pa. 12. of my *Obſervations*; the algebraic ſummation being poſſible when, only $\frac{m}{n}-\frac{p}{q}$, $\frac{t}{u}-\frac{r}{s}$ being *even* numbers, $\frac{p}{q}+\frac{m}{n}$ is $=\frac{r}{s}+\frac{t}{u}$.

Equally aukward, or more ſo, is his formula pa. 96. 97. for the ſum of the ſeries.

$$\frac{a+b.c+e\,x^{r:s+1}}{p+q.m+n.r+s}\pm\frac{a+2b.c+2e.x^{r:s+2}}{p+2q.m+2n.\quad+2s}\ \&c.$$

from which he has deduced ſuch inferences reſpecting the algebraic ſummation thereof, as plainly evince, that he was *completely puzzled by his own proceſs:* the concluſions he has inferred* agreeing only with the ſeries

$$\frac{c+e}{p+q.r+s} - \frac{c+2e}{p+2q.r+2s} + \frac{c+3e}{p+3q.r+3s} - \frac{c+4e}{p+4q.r+4s} \text{ \&c.}$$

and $\frac{a+b}{p+q.m+n} - \frac{a+2b}{p+2q.m+2n} + \frac{a+3b}{p+3q.m+3n} - \frac{a+4b}{p+4q.m+4n}$ &c.

inſtead of the ſeries propoſed: and the examples he has given are equally nugatory!—So that he has not given one example of the *algebraic* ſummation of the Series.

$$\frac{a+b.c+e}{p+q.m+n.r+s} - \frac{a+2b.c+2e}{p+2q.m+2n.r+2s} + \frac{a+3b.c+3e}{p+3q.m+3n.r+3s} \text{ \&c.}$$

but, in attempting to give ſuch examples, has given others foreign to the purpoſe!

Such are the conſequences of Mr. Lorgna's embarraſſed method! And, what is ſtill more extraordinary, the nugatory concluſions juſt now adverted to are mentioned by Mr. Clarke (pa. 51 of his Suppl.) as inſtances that Mr. Lorgna was not ignorant of the algebraic ſummation depending on the different combinations of the fluentials in his general formulæ!—adding, that his not particularizing the caſes of ſuch ſummation in ſome of his other theorems can, at worſt, be called only an inadvertent omiſſion. A pretty excuſe indeed, after his failing where he attempted to particularize ſuch caſes; and yet boaſting of what he had done, with reſpect to pointing out the *criterion* of ſuch ſummation, as a very material improvement in the doctrine! Will it be believed, that if Mr. Lorgna (who, it ſeems, did not aim at brevity) had well underſtood the problem, he would have omitted ſome principal parts of its ſolution? And can it be ſuppoſed that if Mr. Clarke had underſtood it better, he would not in his notes have ſupplied the defects?

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* In conſequence of taking one of the factors in the numerators equal (or in a conſtant ratio) to one of thoſe in the denominators.

In my *Obſervations*, (ſtudying to avoid an unneceſſary increaſe in the length of the pamphlet,) I omitted taking particular notice of Mr. Lorgna's 7th *Section*, thinking I ſhould ſufficiently explain the uſe of Mr. Simpson's theorem without adverting to that ſection. However, to ſatisfy Mr. Clarke, who (pa. 52. of his Suppl.) ſeems to wonder at the omiſſion, I will take this opportunity of ſhewing ſome regard to that neglected ſection: to the purpoſe of which, the theorem at pa. 3. of my *Obſervations* will be adapted if p and q be each $= 2$; A being then $= \frac{c-r.d-r}{s-r.t-r}$, B $= \frac{c-s.d-s}{r-s.t-s}$, C $= \frac{c-t.d-t}{r-t.s-t}$; D, E, &c. each $= o$;

$$\text{and } Q = \frac{cd}{rst} \pm \frac{c+1.d+1.a}{r+1.s+1.t+1} + \frac{c+2.d+2.a^2}{r+2.s+2.t+2} \pm \frac{c+3.d+3.a^3}{r+3.s+3.t+3} \text{ \&c.}$$

$$= fl.\ \frac{\dot{x}}{1 \mp x} \times \overline{\frac{c-r.d-r.x^{r-1}}{s-r.t-r.a^r} + \frac{c-s.d-s.x^{s-1}}{r-s.t-s.a^s} + \frac{c-t.d-t.x^{t-1}}{r-t.s-t.a^t}}$$

generated whilſt x from o becomes $= a$:

which expreſſion* when $s-r$, $t-r$ are poſitive integers, and a is a root of the equation

$$\frac{c-r.d-r.a^{-r}}{s-r.t-r} + \frac{c-s.d-s.a^{-s}}{r-s.t-s} + \frac{c-t.d-t.a^{-t}}{r-t.s-t} = o,$$

will (if finite) be

$$= \frac{c-s.d-s}{s-r.t-s} a^{r-s} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} (s-r)}$$

$$+ \frac{c-t.d-t}{t-r.s-t} a^{r-t} \times \overline{\frac{1}{r} + \frac{a}{r+1} + \frac{a^2}{r+2} (t-r)}$$

If $s-r$, $t-r$ be both *even* numbers the value of Q (which is *infinite* when a is $= 1$) cannot be expreſſed *algebraically* when a is $= -1$.

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* Suppoſing the upper ſign to take place in the denominator $1 \mp x$

But the value of Q may be ſo expreſſed when a is $= -1$, if,
$s-r$, $t-r$ being both *odd* numbers, A be $=\frac{1}{2}$;
or $t-r$ being *even* and $s-r$ *odd*, B be $=\frac{1}{2}$;
or $t-r$ being *odd* and $s-r$ *even*, C be $=\frac{1}{2}$;

Example I. The ſeries propoſed being

$$\frac{4.7}{1.2.3} - \frac{5.11}{2.\ 3.4} + \frac{6.15}{3.\ 4.5} - \frac{7.19}{4.\ 5.6} \text{ \&c.} = Q;$$

we have $c=4$, $d=\frac{7}{4}$, $r=1$, $s=2$, $t=3$, $a=-1$,

and $Q=\frac{13}{4}$.

Example II. If the ſeries whoſe ſum is ſought be

$$\frac{6.7}{1.4.7} - \frac{7.11}{2.\ 5.8} + \frac{8.15}{3.\ 6.9} - \frac{9.19}{4.\ 7.10} \text{ \&c.} = Q;$$

c will be $=6$, $d=\frac{7}{4}$, $r=1$, $s=4$, $t=7$, $a=-1$,

and $Q=\frac{341}{360}$

Example. III. If the ſeries propoſed be

$$\frac{7.9}{1.4.6} - \frac{8.13}{2.\ 5.7} + \frac{9.17}{3.\ 6.8} - \frac{10.\ 21}{4.7.9} \text{ \&c.} = Q;$$

we have $c=7$, $d=\frac{9}{4}$, $r=1$, $s=4$, $t=6$, $a=-1$,

and $Q=\frac{191}{120}$.

Thus, by Mr. SIMPSON's theorem, are ſeries found with two factors in each numerator and three in each denominator, the ſigns being alternately + and −, whoſe ſums are expreſſible *algebraically.* Which ſort of ſeries Mr. LORGNA in vain attempted to find by his ill-expreſſed theorem pa. 96. 97!—Mr. CLARKE likewiſe has been *puzzling himſelf* in vain about the ſame buſineſs at pa. 5. of his *Supplement*; where his particular caſes (even as he himſelf acknowledges) all relate to ſeries with only one factor in each numerator and two factors in each denominator, inſtead of relating to the ſeries under conſideration! Had I conſidered, in my *Obſervations*, the ſeries here conſidered, probably we might not have found his remarks (in his *Supplement)* on particular caſes of that ſeries ſo egregiouſly inſignificant as they now are.

His remarks at pa. 6. and 7. of his Supplement (on particular caſes, about which he has been farther puzzling himſelf) are not leſs ſhallow and inſignificant than thoſe juſt now adverted to. For in the caſe he mentions art. 2. pa. 6. $\frac{a}{b}$ and $\frac{c}{e}$ being reſpectively equal to $\frac{r}{s}$ and $\frac{t}{u}$, his propoſed ſeries $\frac{a+b.c+e}{p+q.m+n.r+s.t+u} \pm \frac{a+2b.c+2e}{p+2q.m+2n.r+2s.t+2u}$ &c. becomes $\frac{1}{p+q.m+n} \pm \frac{1}{p+2q.m+2n}$ &c. and R is $= 0$; of which he ſeems not to have been aware! The like may be ſaid of art. 3, where (in the caſe he falls into) Q is $= 0$! And, by what he ſays at art. 4. pa. 7. his want of a clear underſtanding of the caſes he undertakes to explain is farther betrayed; the ſum being not infinite (as he concludes it is) when, π being $= \delta$, x is $= 1$: nor is it requiſite that π be $= \delta$ (as he imagines) when x has that value! See Ex. 3. towards the end of this *Appendix*.

Another inſtance perhaps of his being puzzled is his repeating at pa. 9. a theorem inſerted at pa. 8. For No. 273 is no other than No. 272, with $b + \frac{1}{2}g + 1$ wrote therein inſtead of b!

Truly may it be ſaid, we ſeldom find a writer ſo little ſkilled in the ſubject he treats of, as our author and commentator appear to have been with reſpect to the claſs of ſeries whoſe ſummation they have attempted!!

Mr. Clarke's argument, that the theorems in my *Obſervations* cannot poſſibly be deduced from what Mr. Simpson has done, without adopting the ſame principles, or method of procedure, as laid down in Mr. Lorgna's Treatiſe, is as frivolous and trifling as his other arguments are. The inveſtigation of the general thereom, with which we are furniſhed by Mr. Simpson, is the primary buſineſs in the ſolution of the problem under conſideration: the adapting that theorem to particular caſes, ſo as to diſcover the *criterion* of algebraic ſummation, and to expreſs the ſum algebraically when it can be ſo expreſſed, is a more eaſy ſecondary buſineſs, which may be performed by diviſion, or the compariſon of fluents. Mr. LORGNA's method, after finding the general formula in terms of fl. $\frac{x^{p:q}\dot{x}}{1\mp x}$, fl. $\frac{x^{m:n}\dot{x}}{1\mp x}$, &c. is to obtain therefrom another expreſſion in terms of the form fl. $\frac{\overline{1-x}^{\pi}\times x^{p:q}\dot{x}}{1\pm x}$, from whence the ſum ſought is found in algebraic terms when $1-x^{\pi}$ is diviſible by $1\mp x$; but this method does not point out the *complete* criterion for all the caſes wherein the algebraic ſummation is poſſible, as the method which I have purſued does. I bring all the terms in which fl. $\frac{x^{m:n}\dot{x}}{1\mp x}$. fl. $\frac{x^{r:s}\dot{x}}{1\mp x}$, &c. are concerned into algebraic terms and terms of fl. $\frac{x^{p:q}\dot{x}}{1\mp x}$; and I get rid of this laſt mentioned fluent by making the ſum of the terms vaniſh into which it is multiplied. Which methods of getting rid of fl. $\frac{x^{p:q}\dot{x}}{1\mp x}$, &c. are clearly very different; though the algebraic terms afterwards retained are the ſame when the caſes are the ſame. The agreement of the concluſions deduced by the two methods, in certain caſes, does not prove that the methods of deducing them are ſimilar. However this ſecondary buſineſs of transforming the general theorem ſo as to diſcover the criterion above mentioned is alſo performed by Mr. SIMPSON, in a peculiar manner, in his 4th propoſition; ſo that nothing remains to be done in that propoſition to adapt it to the purpoſe of finding ſuch criterion by any method of procedure ſimilar to Mr. LORGNA's. And in the very ſame manner I might have deduced my theorems (alluded to by Mr. CLARKE) from what Mr. SIMPSON has done in his 5th propoſition, had I choſe to have done ſo. Whence it follows, that I could have deduced thoſe theorems without adopting the ſame principles, or method of procedure, as laid down in Mr. LORGNA's Treatiſe; unleſs it be admitted that his principles are the ſame as Mr. SIMPSON's, which I think Mr. CLARKE will hardly admit. What then becomes of his firſt objection? He muſt either confeſs, that it is groundleſs, or that Mr. LORGNA (in the buſineſs under conſideration) has followed Mr. SIMPSON in his method of procedure! The weakneſs of his aſſertion is farther remarkable! for, if he had proved, that I have actually adopted Mr. LORGNA's principles, or method of procedure; it would not have followed, that *I cannot poſſibly* do the buſineſs otherwiſe.

I am really at a loſs to know what Mr. CLARKE means by his ſecond objection in his ſupplement. Whatever he might think before my *Obſervations* were publiſhed, I cannot conceive, that he *now* means to aſſert, that the criterion alluded to extends to all the caſes wherein an algebraic ſummation is poſſible, with reſpect to ſuch ſeries as Mr. LORGNA has conſidered. If he would have it underſtood that his author has only in part conſidered the caſes of ſuch ſummation; that is, when there is no given relation ſuppoſed between the factors, as what is ſaid at pa. 51. ſeems to import; I admit that Mr. LORGNA has done no more. That narrow conſideration is the very occaſion of my objecting to what he calls the *indubitable mark*, or *true criterion* of the poſſibility of an algebraic ſummation: which, as he ſays not a word of any defect in it, but (on the contrary) ſpeaks of it (pa. 23) as a method of finding with certainty whether ſuch ſeries be ſummible algebraically or not,* we may reaſonably ſuppoſe he conſidered as quite perfect, and as affording the only rules neceſſary for diſcovering the caſes in which the ſums of any ſeries, of the claſs which he treats of, may be expreſſed algebraically; whereas there are innumerable caſes (as intereſting as thoſe pointed out by his rules) in which the ſums of ſuch ſeries may be ſo expreſſed, that will not be diſcovered by that imperfect mark. Of ſuch caſes I have already given ſome inſtances: and more extenſive rules, for diſcovering when an algebraic ſummation is attainable, are given towards the end of this *Appendix*; whereby it appears that his rules are only particular caſes of others much more general, which are deduced as corollaries from what Mr. SIMPSON has done.

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* "And a method is pointed out how to diſtinguiſh with certainty whether the "given ſeries of this order be ſummible or not." See the *Preface*.

I ſee no reaſon to conclude, that Mr. LORGNA knew more of the ſubject than he has taught us. If he had known more, would not his giving only ſuch imperfect rules, for diſcovering when an algebraic ſummation is poſſible, have been inexcuſable; after obſerving, that he looked upon a rule of that kind as of the greateſt utility in abſtracted mathematics?

Of the numerous errors which I might have remarked in his treatiſe, I have (in my *Obſervations*) only adverted to three; (ſee pa. 8, 10, and 13) which are all of one kind, and are ſuch as I conſider as errors of judgment, (that might, if not corrected, miſlead the reader) not as trifling inaccuracies, or negligencies, as the ſupplement-writer calls them. For, whatever he and his author may *now* know reſpecting the general value of $\frac{1-y^g}{1-y^b}$; it is not to be doubted (after ſo many proofs) that, as they had, in the caſe where $y = 1$, repeatedly conſidered the Log. of $\frac{1-y^g}{1-y^b}$ as the Log. of 1, and therefore $= 0$; if ſuch expreſſion had again and again occurred, they would again and again have committed the ſame error: which, probably, to this time, would have remained uncorrected by them, if it had not been pointed out to them. The conſequences of which, as far as Mr. LORGNA's concluſions are affected by it, are noticed by me, with the ſame ſtrict regard to veracity as is obſerved in the writing down every other article in the pamphlet.

There are a few other articles, in Mr. CLARKE's Supplement, of which I muſt take ſome notice. He calls upon me (ſurely not very pertinently) at pa. 52. to give the reaſon why I have not, in my *Obſervations*, ſummed one ſingle ſeries *algebraically* by my theorems, when the number of factors in the numerators is only *one leſs* than the number of thoſe in the denominators, the ſigns being alternately + and −, and $x = -1$! Will he be candid enough to acknowledge his inadvertance, when he is told, that ſeries of that ſort are ſummed in Ex. 1. and 2. of the 4th Cor. of the work alluded to? Where, by taking $r = 1$, in Ex. 1. we have $\frac{3}{1.2} - \frac{5}{2.3} + \frac{7}{3.4} - \frac{9}{4.5}$ &c. $= 1$; and by taking $k = 3$, in Ex. 2. $\frac{2}{3.5} - \frac{3}{5.7} + \frac{4}{7.9} - \frac{5}{9.11}$ &c. $= \frac{1}{12}$.

What could induce the ſupplement writer to aſk pa. 53.) ‘ what “ Author beſides Mr. LORGNA hath ever ſhewn in a *general manner* “ the impoſſibility of the ſummation of the ſeries

$$“\frac{ab\,(k-1)}{pqr\,(k)} + \frac{a+1.b+1\,(k-1)}{p+1.q+1.r+1\,(k)} \text{ \&c.”}$$

ſeeing that Mr. LORGNA, ſo far from having ſhewn in a *general manner* the impoſſibility of the ſummation of that ſeries, has not even ſhewn the impoſſibility of the ſummation thereof in *any manner* when k is greater than 2? Indeed at pa. 100 he ſays “ *it might be* eaſily demonſtrated, if “ neceſſary, that theſe ſeries having three factors in the denominators “ (that is when the greateſt power of the index z in the numerator “ is but one degree lower than the greateſt power of the ſame quanti- “ ty in the denominator) cannot poſſibly be ſummed when the ſigns “ are poſitive; the general expreſſion for the ſum in that caſe involv- “ ing a quantity abſolutely infinite.” But is this to be called demonſtration? The general expreſſion for the ſum does then, it is true, involve an infinite quantity, and the ſum is infinite; but this does not immediately appear from Mr. Lorgna's formula. Nor has he done *any thing* reſpecting the proof of impoſſibility when k is greater than 3! For any thing that appears in his book, the coefficient of the infinite quantity might, in certain caſes, be $=0$; and conſequently, that quantity vaniſhing, the remainder of the expreſſion for the ſum might be finite.

From the conſideration of Mr. Simpson's proceſs, it appears, that $(A+B+C(k))$ the coefficient of the infinite part of the expreſſion for the ſum will always be $=1$, when $k-h$ is $=1$; and conſequently, the ſum will then be infinite.

Inadvertence ſurely cannot account for a queſtion ſo little to the purpoſe!

Another queſtion, not leſs unaccountable nor more to the purpoſe, than the preceding, the Supplement-writer has alſo called upon me to anſwer! He aſks, who, beſides his favourite author, hath ever ſhewn in a *general manner* the conditions or neceſſary relation of the factors for the algebraic ſummation of the ſeries

$$\frac{ab(k-1)}{pqr(k)} - \frac{a+1.b+1.(k-1)}{p+1.q+1.r+1(k)} + \frac{a+2.b+2.(k-1)}{p+2.q+2.r+2(k)} - \frac{a+3.b+3(k-1)}{p+3.q+3.r+3(k)} \text{ \&c. ?}$$

though (unfortunately) it does not appear, that Mr. Lorgna has given any *general* theorem reſpecting the ſummation of this ſeries!

But from Mr. Simpson's *general* theorem, we have the expreſſion for the ſum, let the factors be what they may; and their neceſſary relation, that the ſummation may be *algebraic*, is ſhewn by the equation ariſing upon conſidering the coefficient of S as $=0$, z^n being taken $=-1$.

The following remarkable corollaries (relative to the matter in queſtion) are deducible from Mr. Simpson's ſaid theorem, and the conſideration of his *Lemma* (pa. 76): and, as neither Mr. LORGNA, nor Mr. CLARKE, has given us the like theorems, I preſume they may not be unacceptable to the intelligent reader: who will find, that they exhibit very comprehenſive rules for diſcovering the caſes of algebraic ſummation; and give the expreſſion for the ſum of the ſeries, when it appears by thoſe rules, that ſuch ſummation is poſſible.

COR. I. When the differences $\overset{\prime\prime}{p}-p,\ \overset{\prime\prime\prime}{p}-p,\ \overset{iv}{p}-p,\ (k-1)$ are all *even* numbers,

$$Q=\frac{ab(k-1)}{p\overset{\prime\prime}{p}\overset{\prime\prime\prime}{p}\,(k)}-\frac{a+1\ b+1(k-1)}{p+1.\overset{\prime\prime}{p}+1.\overset{\prime\prime\prime}{p}+1(k)}+\frac{a+2.b+2(k-1)}{p+2.\overset{\prime\prime}{p}+2.\overset{\prime\prime\prime}{p}+2(k)}-\&c.$$

will be $=fl.\ \frac{z^{p-1}\dot{z}}{1+z}$, generated whilſt z from o becomes = 1,

$$-\overset{\prime}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\frac{1}{p+3}\,(\overset{\prime\prime}{p}-p)}$$

$$-\overset{\prime\prime\prime}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\frac{1}{p+3}(\overset{\prime\prime\prime}{p}-p)}$$

$$(k-1)$$

$$\overset{\prime}{A}\text{ being }=\frac{a-p.b-p(k-1)}{\overset{\prime\prime}{p}-p.\overset{\prime\prime\prime}{p}-p(k-1)},\quad \overset{\prime\prime}{A}=\frac{a-\overset{\prime\prime}{p}.b-\overset{\prime\prime}{p}\,(k-1)}{p-\overset{\prime\prime}{p}.\overset{\prime\prime\prime}{p}-\overset{\prime\prime}{p}(k-1)},$$

$$\overset{\prime\prime\prime}{A}\quad=\frac{a-\overset{\prime\prime\prime}{p}.b-\overset{\prime\prime\prime}{p}(k-1)}{\overset{\prime\prime\prime\prime}{p}-\overset{\prime\prime}{p}.\overset{\prime\prime\prime}{p}-p(k-1)},\ (k).$$

whence it is evident that the value of Q, in ſuch caſe, cannot be expreſſed *algebraically*.

But, *m* being leſs than $k-1$, if the differences $\overset{\prime\prime}{p}-p,\ \overset{\prime\prime\prime}{p}-p,\ \overset{iv}{p}-p,(m)$ be *even* numbers, and the reſt of the like differences be *odd* numbers,

$$Q\text{ will be}=\overline{2\overset{\prime}{A}+2\overset{\prime\prime}{A}+2\overset{\prime\prime\prime}{A}\,(m+1)-1}\times fl.\ \frac{z^{p-1}\dot{z}}{1+z},$$

generated whilſt z from o becomes = 1,

– A

$$-\overset{\prime\prime}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\frac{1}{p+3}\,(\overset{\prime\prime}{p}-p)}$$

$$-\overset{\prime\prime\prime}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\frac{1}{p+3}\,(\overset{\prime\prime\prime}{p}-p)}$$

$$(m)$$

$$+\overset{(m+2)}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\frac{1}{p+3}\,(\overset{(m+2)}{p}-p)}$$

$$+\overset{(m+3)}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\frac{1}{p+3}\,(\overset{(m+3)}{p}-p)}$$

$$(k-m-1).$$

Conſequently, if $\overset{\prime}{A}+\overset{\prime\prime}{A}+\overset{\prime\prime\prime}{A}\,(m+1)$ be $=\frac{1}{2}$;
or, which comes to the ſame thing,

$$\overset{(m+2)}{A}+\overset{(m+3)}{A}+\overset{(m+4)}{A}\,(k-m-1)\text{ be }=\tfrac{1}{2};$$

the value of Q will be expreſſed *algebraically* : – whereof I have already given examples (pa. 13).

Cor. II. If, b being leſs than $k-1$, the differences $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$, $\overset{iv}{p}-p$, (m) be integers, and the reſt of the like differences be integers, or fractions whoſe differences are integers; and

$$\overset{\prime}{A}+\overset{\prime\prime}{A}+\overset{\prime\prime\prime}{A}\,(m+1)\text{ be }=0;$$

or, which comes to the ſame thing,

$$\overset{(m+2)}{A}+\overset{(m+3)}{A}+\overset{(m+4)}{A}\,(k-m-1)\text{ be }=0;$$

the ſummation will be *algebraical*,

$$Q=\frac{a\,b\,(b)}{p\,\overset{\prime\prime}{p}\,\overset{\prime\prime\prime}{p}\,(k)}+\frac{a+1.b+1\,(b)}{p+1.\overset{\prime\prime}{p}+1.\overset{\prime\prime\prime}{p}+1\,(k)}+\frac{a+2.b+2\,(b)}{p+2.\overset{\prime\prime}{p}+2.\overset{\prime\prime\prime}{p}+2\,(k)}\ \&c.$$

$$\text{being then }=-\overset{\prime\prime}{A}\times\overline{\frac{1}{p}+\frac{1}{p+1}+\frac{1}{p+2}\,(\overset{\prime\prime}{p}-p)}$$

$$-\overset{\prime\prime\prime}{A}\times\overline{\frac{1}{p}+\frac{1}{p+1}+\frac{1}{p+2}\,(\overset{\prime\prime\prime}{p}-p)}$$

$$(m)$$

$$-\overset{(m+3)}{A}\times\overline{\frac{1}{\overset{(m+2)}{p}}+\frac{1}{\overset{(m+2)}{p}+1}+\frac{1}{\overset{(m+2)}{p}+2}\left(\overset{(m+3)}{p}-\overset{(m+2)}{p}\right)}$$

$$-\overset{(m+4)}{A}\times\overline{\frac{1}{\overset{(m+2)}{p}}+\frac{1}{\overset{(m+2)}{p}+1}+\frac{1}{\overset{(m+2)}{p}+2}\left(\overset{(m+4)}{p}-\overset{(m+2)}{p}\right)}$$

$$(k-m-2):$$

$$\overset{\prime}{A}\text{ being}=\frac{a-p.\,b-p\,(b)}{\overset{\prime\prime}{p}-p.\,\overset{\prime\prime\prime}{p}-p(k-1)},\ \overset{\prime\prime}{A}=\frac{a-\overset{\prime\prime}{p}.\,b-\overset{\prime\prime}{p}(b)}{\overset{\prime\prime}{p}-p,\,\overset{\prime\prime\prime}{p}-\overset{\prime\prime}{p}(k-1)},$$

$$\overset{\prime\prime\prime}{A}=\frac{a-\overset{\prime\prime\prime}{p}\,b-\overset{\prime\prime\prime}{p}\,(b)}{\overset{\prime\prime\prime}{p}-\overset{\prime\prime}{p}.\,p-\overset{\prime\prime\prime}{p}(k-1)},\ (k).$$

Cor. III. If, b being leſs than $k-1$, the differences $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$, $\overset{iv}{p}-p$, (m) be *even* numbers, and the reſt of the like differences be any numbers, whole or fractional, whoſe differences are all *even* numbers; and

$$\overset{\prime}{A}+\overset{\prime\prime}{A}+\overset{\prime\prime\prime}{A}\,(m+1)\text{ be}=0;$$

or, which comes to the ſame thing,

$$\overset{(m+2)}{A}+\overset{(m+3)}{A}+\overset{(m+4)}{A}\,(k-m-1)\text{ be}=0;$$

the ſummation will likewiſe be *algebraical*,

$$\overset{\prime\prime}{Q}=\frac{ab\,(b)}{p\,\overset{\prime\prime}{p}\,\overset{\prime\prime\prime}{p}\,(k)}-\frac{a+1.\,b+1.\,(b)}{p+1.\overset{\prime\prime}{p}+1.\overset{\prime\prime\prime}{p}+1(k)}+\frac{a+2.\,b+2\,(b)}{p+2.\overset{\prime\prime}{p}+2.\overset{\prime\prime\prime}{p}+2.(k)}-\&c.$$

$$\text{being then}=-\overset{\prime\prime}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\left(\overset{\prime\prime}{p}-p\right)}$$

$$-\overset{\prime\prime\prime}{A}\times\overline{\frac{1}{p}-\frac{1}{p+1}+\frac{1}{p+2}-\left(\overset{\prime\prime\prime}{p}-p\right)}$$

$$(m)$$

$$-\overset{(m+3)}{A}\times\overline{\frac{1}{\overset{(m+2)}{p}}-\frac{1}{\overset{(m+2)}{p}+1}+\frac{1}{\overset{(m+2)}{p}+2}-\left(\overset{(m+3)}{p}-\overset{(m+2)}{p}\right)}$$

$$-\overset{(m+4)}{A}\times\overline{\frac{1}{\overset{(m+2)}{p}}-\frac{1}{\overset{(m+2)}{p}\;+1}+\frac{1}{\overset{(m+2)}{p}\;+2}-\left(\overset{(m+4)}{p}-\overset{(m+2)}{p}\right)}$$

$$(k-m-2):$$

$\overset{\prime}{A}$, $\overset{\prime\prime}{A}$, $\overset{\prime\prime\prime}{A}$, &c. being as in the preceding corollary.

Example I. If, r and t being integers, $2q+r$ be $=2s+t$; the ſeries

$$\frac{1}{q.\overline{q+r}.s.\overline{s+t}}\pm\frac{1}{\overline{q+1}.\overline{q+r+1}.\overline{s+1}.\overline{s+t+1}}+$$

$$\frac{1}{\overline{q+2}.\overline{q+r+2}.\overline{s+2}.\overline{s+t+2}}\pm\text{\&c.}$$

will be $=\dfrac{1}{r.\overline{q-s}.\overline{s-q-r}}\times\overline{\dfrac{1}{q}\pm\dfrac{1}{q+1}+\dfrac{1}{q+2}\pm\dfrac{1}{q+3}\,(r)}$

$$+\frac{1}{t.\overline{s-q}.\overline{s-q-r}}\times\overline{\frac{1}{s}\pm\frac{1}{s+1}+\frac{1}{s+2}\pm\frac{1}{s+3}\,(t)}:$$

but, with reſpect to the *lower* of the double ſigns, the theorem only holds true when r and t are both *even* numbers.—See *Obſervat.* pa. 12. and 28.

Example II. If, r and t being integers, $\overline{s-q}\times\overline{s+t-q}$ be $=\overline{a-q}\times\overline{2s+t-2}\,\overline{q-r}$; the ſeries

$$\frac{a}{q.\overline{q+r}.s.\overline{s+t}}\pm\frac{a+1}{\overline{q+1}.\overline{q+r+1}.\overline{s+1}.\overline{s+t+1}}+$$

$$\frac{a+2}{\overline{q+2}.\overline{q+r+2}.\overline{s+2}.\overline{s+t+2}}\pm\text{\&c.}$$

will be $=\dfrac{a-q-r}{r.\overline{s-q-r}.\overline{s+t-q-r}}\times\overline{\dfrac{1}{q}\pm\dfrac{1}{q+1}+\dfrac{1}{q+2}\pm\dfrac{1}{q+3}\,(r)}$

$$+\frac{a-s-t}{t.\overline{q-s-t}.\overline{q+r-s-t}}\times\overline{\frac{1}{s}\pm\frac{1}{s+1}+\frac{1}{s+2}\pm\frac{1}{s+3}\,(t)}:$$

but, with reſpect to the *lower* of the double ſigns, the theorem only holds true when r and t are both *even* numbers.

The ſame may be found from Cor. 1. of my *Obſervat.* by taking the difference of two ſuch ſeries as are therein conſidered.

It is obſervable that, if t be $= r$, and a be ($= \frac{1}{2} \times \overline{q+r+s}$) equal to half the ſum of the greateſt and leaſt factors in the denominator of the firſt term, the expreſſion for the ſum of the ſeries becomes

$$= \frac{1}{2r} \times \overline{\frac{1}{qs} \pm \frac{1}{q+1.s+1} + \frac{1}{q+2.s+2} \pm \frac{1}{q+3.s+3}}\,(r).$$

Example III. If, r and t being integers, $\overline{a+b-2q-r} \times \overline{s-q} \times \overline{s+t-q}$ be $= \overline{a-q} \times \overline{b-q} \times \overline{2s+t-2q-r}$; the ſeries

$$\frac{ab}{q.q+r.s.s+t} \pm \frac{a+1.b+1}{q+1.q+r+1.s+1.s+t+1} + \frac{a+2.b+2}{q+2.q+r+2.s+2.s+t+2} \pm \&c.$$

will be $= \frac{a-q-r.b-q-r}{r.s-q-r.s+t-q-r} \times \overline{\frac{1}{q} \pm \frac{1}{q+1} + \frac{1}{q+2} \pm \frac{1}{q+3}}\,(r)$

$$+ \frac{a-s-t.b-s-t}{t.q-s-t.q+r-s-t} \times \overline{\frac{1}{s} \pm \frac{1}{s+1} + \frac{1}{s+2} \pm \frac{1}{s+3}}\,(t):$$

but, with reſpect to the *lower* of the double ſigns, the theorem only holds true when r and t are both *even* numbers.

It may be obſerved, that, whenever the ſum of the ſeries in the firſt example can be expreſſed algebraically, as in that example; the ſum of the ſeries in this example may be ſo expreſſed, in the terms here exhibited, if $a+b$ be ($= 2q+r = 2s+t$) equal to the ſum of the greateſt and leaſt factors in the denominator of the firſt term of the ſeries. Which is the moſt remarkable, but not the only, caſe to which this theorem extends.

By taking $q = b - \frac{1}{2}$, $r = 2$, $s = b$, $t = 1$, and a, b, each $= b + \frac{1}{2}$; we find the ſum of the ſeries

$$\frac{\overline{2b+1}|^2}{b.b+1.2b-1.2b+3} + \frac{\overline{2b+3}|^2}{b+1.b+2.2b+1.2b+5} + \frac{\overline{2b+5}|^2}{b+2.b+3.2b+3.2b+7}\ \&c.$$

$$= \frac{1}{3} \times \frac{12b^2+1}{b.2b-1.2b+1}.$$

Which ſum is to the ſum of the ſeries whoſe denominators are the ſame as in this ſeries, and whoſe numerators are each $=1$, as $12b^2+1$ to 1.—See *Obſervat.* pa. 28.

And by taking $r=4$, $s=q+1$, $t=2$, and a, b each $=q+2$; we have

$$\frac{\overline{q+2}|^2}{q\cdot q+1\cdot q+3\cdot q+4} \pm \frac{\overline{q+3}|^2}{q+1\cdot q+2\cdot q+4\cdot q+5} + \frac{\overline{q+4}|^2}{q+2\cdot q+3\cdot q+5\cdot q+6} \pm \&c.$$

$$= \tfrac{1}{6} \times \frac{\overline{2q+3} \times \overline{3q^2+9q+4}}{q\cdot q+1\cdot q+2\cdot q+3}, \text{or} = \tfrac{1}{2} \times \frac{q^2+3q+4}{q\cdot q+1\cdot q+2\cdot q+3};$$

according as the ſigns are all poſitive, or alternately poſitive and negative.

Which ſum is to the ſum of the ſeries whoſe denominators are the ſame as in this ſeries, and whoſe numerators are each $=1$, as $3q^2+9q+4$ to 1, or as q^2+3q+4 to 1; according as the ſigns are all poſitive, or alternately poſitive and negative. See *Obſervat.* pa. 12.

When m is $=k-1$, the two laſt corollaries coincide with Mr. Lorgna's rules, as far as his rules extend.

Having ſo fully explained the ſubject which I propoſed to diſcuſs, I now ſubmit it to the public to determine, whether Mr. Clarke has not *without reaſon* extolled the production of a *foreigner*; and diſingenuouſly *depreciated* the work of our *countryman*, by imputing (particularly in his *Preface* to Mr. Lorgna's book) impropriety and narrowneſs to Mr. Simpson's method of inveſtigation; and, with reſpect to perſpicuity and generality, giving the preference to Mr. Lorgna's, which is plainly ſhewn to be ſo embarraſſed as to have puzzled not only the *Author himſelf*, but the *Commentator* alſo!

It may not be improper, before I conclude, to caution Mr. Clarke to have a little more regard to propriety; and not to write in ſuch a random manner! With him, every thing of Mr. Lorgna's is wonderful! ſeries are ſummed whoſe ſummation had always been thought impoſſible, though comprehended in a theorem publiſhed near *forty* *years* ago; proof in a particular caſe, or two, is general demonſtration; perplexity is perſpicuity; and aukwardneſs is elegance!

We are told, that he has brought the doctrine of ſeries to its greateſt perfection, and left nothing undone therein that could be expected or wiſhed for! I wonder what *great work* "of thoſe that are ſtill reſerved "for the ſagacity and ſurprizing penetration of a LORGNA"!) he will next produce; that may give farther ſcope for the employment of his friend's extraordinary talent at *book-eking*, or writing notes explanatory and critical.

THE END OF THE APPENDIX.

ERRATA.

In the Obſervations.

Pa. 9, l. 5, *for* $\frac{1}{s-r.t-s}$, *read* $\frac{a^{r-s}}{s-r.t-s}$;

and *for* $\frac{1}{r-t.s-t}$, *read* $\frac{a^{r-t}}{r-t.s-t}$.

Pa. 24, l. 15, *for* $r-1.r\,a^{2}$, *read* $r-1.r-2.a^{2}$;

and *for* $r-1.r.r+1.a^{3}$, *read* $r-1.r-2.r-3.a^{3}$.

POSTSCRIPT.

POSTSCRIPT.

THE value of $\overset{\prime\prime}{S}$ may be found by the equation between the two values of $\overset{\prime\prime}{H}$ at pa. 77. of my *Memoirs*, by taking $x=o$, and conſequently $u=\sqrt{-1}$; though Mr. Clarke, as it ſeems by pa. 22. of his Supplement, could not perceive the poſſibility of finding it by ſuch means!

At pa. 138. of the *Appendix* to my *Memoirs*, the exponents of z are printed n, $n-1$, $n-3$, $n-4$, &c. inſtead of n, $n-1$, $n-2$, $n-3$, &c ſuch miſtakes the *candid* reader will readily correct and excuſe. This is the error which Mr. Clarke alludes to, in his note pa. 20. of his Suppl. leaving the reader to imagine it a very great one!

With reſpect to what may have been diſcovered by *John Francis de Tuſchis de Fagnano* and *his Father*; I can, with the ſtricteſt regard to truth, averr, that I had never, by any means whatever, ſeen (or heard of) any of their Works before the publication of Mr. Clarke's *Supplement*: nor have I ever ſeen any of the *Acta Eruditorum*, or *Diarium Eruditorum Italiæ*, or *the Works printed at Peſaro*, mentioned in that pamphlet.—Its author therefore may take ſhame to himſelf for the illiberality of his reflections!—If I have proved his *want of ſkill*, he himſelf has proved his *want of candour*.

A
SUPPLEMENT
TO
OBSERVATIONS
ON
CONVERGING SERIES.

THE *Obſervations* which I firſt publiſhed being chiefly intended to vindicate the reputation of my deceaſed friend Mr. SIMPSON, by ſhewing, that his method, ſo far from being chargeable with *impropriety* and *narrowneſs*, (which are unjuſtly imputed to it in Mr. CLARKE's *Preface*,) is more elegant, more perſpicuous, and not leſs (if not more) general than Mr. LORGNA's; I omitted many proofs, which I might have adduced from other authors, to ſhew, that the laſt mentioned gentleman's work is, as I had aſſerted, extolled *without reaſon*; being then ſo far from having any intention of criticiſing the tranſlation of that work, and the tranſlator's notes upon it, with the ſeverity they deſerve, that I had no thoughts of taking any farther notice of them. But the Remarks ſince publiſhed by the commentator, in his *Supplement*, led me to write an *Appendix* to thoſe *firſt Obſervations*; and his *Additional Remarks* now lead me to expatiate a little farther on the ſubject, to take ſome notice of what Mr. DE MOIVRE and Mr. STIRLING have done reſpecting the ſame claſs of ſeries as is conſidered by Mr. LORGNA, and to add a few animadverſions on thoſe notable publications of his bungling *book-eker*.

My *Obſervations* were not publiſhed until about *two years* had elapſed (as appears by the dates) after the appearance of the work which occaſioned my writing them.—Time enough ſurely for drawing out *corrections*, and dreſſing up a *Supplement!* For though the numerous errors in that work required not a little time to correct them; yet certainly little or no time could be wanted to make *additions* to a work, which, according to the commentator's own declaration, had brought the doctrine of ſeries to its greateſt perfection, and left nothing undone therein that could be expected or wiſhed for.—But it ſeems *another year* was found neceſſary to hatch the embryo, and bring forth the extraordinary production alluded to! Its appearance, after all, as I had reaſon to conclude would be the reſult, did not evince, that my *Obſervations* were unneceſſary.

I am always ready to acknowledge the merits of any *ingenious* work, whereof I can form any judgment; whether it be the production of a *Britiſh*, or a *foreign* author; of which, I preſume, my publications afford ſome proofs. And, as I am pleaſed when I meet with any thing worthy of applauſe; ſo am I provoked when I ſee a very valuable work diſingenuouſly depreciated, to make it appear inferior to another of little value.

Moved by ſuch provocation, I took up my pen to expoſe the diſingenuity glaring in the work adverted to above: and, I truſt, the public will not conſider me as having (by ſo doing) unfairly attempted to detract from *real* merit; but rather as having laudably endeavoured to prevent the miſrepreſentations of an oſtentatious commentator from operating as detraction to the prejudice of the praiſe-worthy.

Mr. De Moivre (in his *Miſcell. Analyt.* pa. 110) beginning with an eaſy caſe, puts

$$\frac{\dot{x}}{1-x} = \dot{x} \times \overline{1 + x + x^2 \text{ \&c.}} = \dot{v};$$

whence, by taking the fluents, he finds

$$\frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} \text{ \&c.} = vx + x - v.$$

Likewiſe, from the laſt equation, by multiplying by $\dot{x}$ and again taking the fluents, he finds

$$\frac{x^3}{1.2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{3.4.5} \text{ \&c.} = \frac{vx^2}{2} - vx + \frac{v}{2} - \frac{x}{2} + \frac{3x^2}{4}:$$

hence, by taking $x = 1$, it appears that

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} \text{ \&c. is} = \frac{1}{4}.$$

Moreover, from the equation $\dot{x} + x^3\dot{x} + x^6\dot{x}$ &c. $= \frac{\dot{x}}{1-x^3}$, the ſame author, by taking the fluents, gets

$$x + \frac{x^4}{4} + \frac{x^7}{7} \text{ \&c.} = \text{fl.} \frac{\dot{x}}{1-x^3}, = v = \frac{1}{3} \text{ Log. } \frac{1}{1-x} + \frac{2z}{3} + \frac{1}{6} \text{ Log. } \overline{1+x+x},$$

z denoting the circ. arc whoſe rad. is $\sqrt{\frac{3}{4}}$ and tang. $\frac{\frac{3}{4}x}{1+\frac{1}{2}x}$.

Whence, by multiplying by $\dot{x}$ and again taking the fluents, he finds

$$\frac{x^2}{1.2} + \frac{x^5}{4.5} + \frac{x^8}{7.8} \text{ \&c.} = \text{fl. } v\dot{x} = vx - \text{fl. } \dot{v}x = vx - \frac{1}{3} \text{ Log. } \frac{1}{1-x} + \frac{2z}{3}$$

$$- \frac{1}{6} \text{ Log. } \overline{1+x+x^2} = \frac{x-1}{3} \text{ Log. } \frac{\sqrt{1+x+x^2}}{1-x} + \frac{2x+2}{3}z, \text{ fl. } \dot{v}x =$$

$$\text{fl. } \frac{x\dot{x}}{1-x^3} \text{ being} = \frac{1}{3} \text{ Log. } \frac{1}{1-x} - \frac{2z}{3} + \frac{1}{6} \text{ Log. } \overline{1+x+x^2}.$$

It is farther obſerved by Mr. De Moivre (pa. 120, 121) that ſeries conſiſting of terms having ſuch numerators (as well as ſuch denominators) as thoſe belonging to any of Mr. Lorgna's ſeries may be ſummed by taking fluents as above, and by taking the fluxions after multiplying by certain powers of *x*.

Theſe ſteps plainly point out the way to be purſued generally. And by comparing this ſhort view of Mr. De Moivre's method with that given of Mr. Lorgna's, pa. 2, 3, 4, of my *Appendix*, the ſimilarity of the two methods (with reſpect to their *primary* and *fundamental* parts) will, I preſume, be very evident; the one being only more generally and diffuſely applied than the other, with the terms expreſſing the value of the ſeries differently arranged : even the reſolving the term *ſl*.$\dot{v}x$ into the two terms $vx -$ *ſl*.$\dot{v}x$, as Mr. De Moivre has done, is the very ſame buſineſs as is performed by means of Mr. Lorgna's *Lemma*, pa. 16.

With what propriety the applying the method more diffuſely, with the terms expreſſing the value of the ſeries differently arranged, can be denominated a method *entirely new*, the reader will judge.

The point in diſpute is not as the commentator would repreſent it in his *Additional Remarks*, whether his author's *Œcumenical Formulæ* will give the ſums of ſeries in all caſes to which they can be adapted, and by management afford other rules, reſpecting the *algebraic* ſummation, beſides thoſe laid down in his treatiſe; but the point is, has he *already* laid down in that treatiſe rules whereby we may know, before we ſet about computing the ſum of any ſeries by any of his formulæ, whether the caſe be ſuch that it will be obtained in finite *algebraic* terms; and, if it will be *ſo* obtained, to exhibit its value by a conciſe expreſſion adapted to ſuch caſe, that we may not, by recurring to the *Œcumenical Formulæ*, be at the unneceſſary trouble of computing the values of terms, which in the end may deſtroy each other?

And, with regard to that point, I have clearly ſhewn, that Mr. Lorgna has not given us any other than very imperfect rules for that purpoſe in his 4th, 7th, and 8th ſections.

To ſhew now, as the commentator does in his *Addit. Rem.* after having ſeen my improvements, that better rules than thoſe laid down by his author, for the purpoſe abovementioned, might have been deduced from his theorems, is, in a great meaſure, proving the charge which I have brought againſt the tranſlation and notes alluded to; and evincing, that both thoſe gentlemen were, before, either *puzzled*, or *inexcuſably remiſs* in not giving more perfect rules than are to be found in that work, ſeeing that rules of that kind are acknowledged to be of the greateſt utility in this branch of mathematics.

The new ones, which I have given for the ſame purpoſe, towards the end of my *Appendix*, are true in every caſe to which they refer, even in the caſes where Mr. CLARKE (in his *Addit. Rem.*) ſays they are not ſo!

Intelligent readers, for whom thoſe new rules, or theorems, are intended, will, without farther explanation, readily apply them; and will, I truſt, find them not inaccurate: whilſt readers of an inferior claſs may poſſibly miſunderſtand them, and imagine they are erroneous. —A reader of ſuch claſs may perhaps not obſerve, that, in applying the theorem at pa. 20 of my *Appendix*, $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$, $\overset{\prime\prime\prime\prime}{p}-p$, (m) muſt be *even* numbers; and, not regarding that condition, he may, in ſumming ſuch a ſeries as $\frac{5.7}{1.2.3}-\frac{8.9}{2.3.4}+\frac{11.11}{3.4.5}-\frac{14.13}{4.5.6}$ &c. by that theorem, blunderingly take p, $\overset{\prime\prime}{p}$, $\overset{\prime\prime\prime}{p}$ equal to 1, 2, 3 reſpectively, and ſo have $\overset{\prime\prime}{p}-p$ an *odd* number; (ſee Mr. CLARKE's *Addit. Rem.* pa. 12); though the theorem plainly indicates that $\overset{\prime\prime}{p}-p$ muſt be an *even* number: and, in conſequence of ſuch blunder, he may imagine he has found a defect in the theorem, when in reality the defect is only in his own underſtanding!—The theorem rightly applied will indeed appear to give the ſum of the ſeries $\frac{5.7}{1.3.2}-\frac{8.9}{2.4.3}+\frac{11\ 11}{3.5.4}-\frac{14.13}{4.6.5}$ &c. which I venture to pronounce, may be ſafely aſſumed as equal to the preceding ſeries.

If ſuch blunder be avoided, I am pretty confident, none of thoſe defects will be found in the new theorems in my *Appendix* which our cavilling remarker, wiſhing to depreciate them, pleaſed himſelf with imagining he had diſcovered.

If, in applying my theorem (*Append.* pa. 20) to the above ſeries, a be taken $=\frac{5}{3}$, $b=\frac{7}{2}$, $p=1$, $\overset{\prime\prime}{p}=3$, and $\overset{\prime\prime\prime}{p}=2$; we ſhall have $\acute{A}=\frac{5}{6}$, $\overset{\prime\prime}{A}=-\frac{1}{3}$, $\overset{\prime\prime\prime}{A}=\frac{1}{2}$, and the ſum ſought $=4$: not negative as the blundering remarker brings it out!

That the differences $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$, $\overset{\prime\prime\prime\prime}{p}-p$, (m) are to expreſs ***even*** numbers, is evident, not only from the words (or tenor) of the condition in the theorem, but alſo from the ſigns in the algebraic part of the value of the ſeries.

I have, at pa. 14 of my *Obſervations*, ſhewn that

$$Q=\frac{c}{rs}+\frac{c+1.a}{r+1.s+1}+\frac{c+2.a^2}{r+2.s+2}\ \&c.$$

$$\text{will be }=\frac{c-s}{s-r}a^{r-s}\times\overline{\frac{1}{r}+\frac{a}{r+1}+\frac{a^2}{r+2}\ (s-r)},$$

when, $s-r$ being a poſitive integer, $a^{s-r}-\frac{c-s}{c-r}$ is $=0$.

Which theorem, when the value of Q is finite, muſt, under thoſe reſtrictions, be true, let a be what it may.

The value of $fl.\ \frac{x^{r-1}\dot{x}}{1-x}$ is here not at all concerned. Were it required in the general expreſſion from whence this theorem is derived, it might as well be conceived to be generated whilſt x from 0 becomes $=-1$, as whilſt x from 0 becomes $=1$: in the former caſe x^{-r} $fl.\frac{x^{r-1}\dot{x}}{1-x}$ is $=y^{-r}fl.\ \frac{y^{r-1}\dot{y}}{1+y}$ generated whilſt y from 0 becomes $=1$; y being ſubſtituted for $-x$. The double ſigns, in the ſeries and its value, are therefore not abſolutely neceſſary.

If a be $= -1$, it is plain $s-r$ muſt be an *odd* number; otherwiſe s muſt be $= r$, and the theorem will fail, becauſe we cannot then aſſign the value of the expreſſion for the ſum. But, a being any quantity greater than -1 but leſs than 1, it is alſo plain that $s-r$ may denote either an *even* or an *odd* number. Therefore, when $s-r$ is an *odd* number, a may be any quantity leſs than 1 that is not leſs than -1. In Ex. 1 and 2, pa. 14, $s-r$ is an *odd* number; and thoſe examples are evidently contrived to correſpond with the ſeries $\frac{3}{1.2} - \frac{5}{2.3} + \frac{7}{3.4} -$ &c. and $\frac{2}{3.5} - \frac{3}{5.7} + \frac{4}{7.9} -$ &c. propoſed Ex. 1 and 3, pa. 73, of Mr. CLARKE'*s tranſlation*, upon taking $r=1$ in the former, and $k=3$ in the latter: as is Ex. 3 (*Obſervat.* pa. 15) to correſpond with Ex. 3 (*Tranſlat.* pa. 70), by taking $r=2$. Yet that gentleman has ſo little diſcernment, or ſo much malapertneſs, as to tell me, I have not there ſummed the ſeries Q when a is $= -1$! though it is manifeſt, that, to ſum ſuch ſeries, it is only requiſite to take a therein $= -1$, and to obſerve the reſtrictions in the theorem as ſtated above.—Surely ſomething worſe than *inadvertence* is obſervable in our remarker's miſrepreſentations of facts, ſo far as they relate to paſſages in my writings!

Readers, who underſtand the ſubject, will obſerve, that I have given the value of that ſeries when a is $= -1$, or any greater quantity leſs than 1, by *one* elegant theorem; whilſt Mr. LORGNA, after a long laboured proceſs, to effect the ſame buſineſs, has *aukwardly* made uſe of *two* theorems for the different caſes! And ſuch readers will be aſtoniſhed at the commentator's effrontery, when they obſerve (as they may) that he has even aimed to give, at pa. 4 of his *Suppl.* a theorem tantamount to mine, as an improvement upon what his author had ſo *aukwardly* done!

My method of writing a ſeries that may be ſummed in algebraic terms, in ſuch a manner as to comprehend many numerical ſeries, and to anſwer, in each of them, the purpoſe of Mr. CLARKE'*s ſummatrix*, will, whatever he may think of it, be approved, I preſume, by every candid reader. And I almoſt aſſure myſelf, it will be allowed by ſuch reader, that my notation is preferable to Mr. Lorgna's; being, with *half* the characters he makes uſe of, as general as his, and as eaſily adapted to particular caſes.

Many indeed are the notable devices which the commentator has made uſe of, to make his author appear ſuperior to every other, with regard to the ſummation of ſeries! I will here ſelect one or two, which perhaps may be reckoned curious.

At pa. 45, 46 of the tranſlation under conſideration, the following ſeries are propoſed to be ſummed, *viz.*

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\&c.\ (=A),$$

$$\frac{1}{2.6.10}-\frac{1}{4.8.12}+\frac{1}{6.10.14}-\&c.\ (=B),$$

$$\frac{1}{1.9.10}-\frac{1}{2.12.12}+\frac{1}{3.15.14}-\&c.\ (=C),$$

$$\frac{1}{2.6.8}+\frac{1}{3.8.10}+\frac{1}{4.10.12}+\&c.\ (=D).$$

Upon which, to catch what he thought a favourable opportunity for introducing a *puff* or two, the commentator remarks (pa. 55), that, with reſpect to the firſt of thoſe ſeries, "we cannot but obſerve how "much more elegant and conciſe this (Mr. Lorgna's) method of "ſummation is than that given by De Moivre in his *Miſcell. Analyt.* "pa. 114, for the ſame ſeries." (See pa. 3 of this *Supplement* and judge for yourſelf good reader.) And he adds, "it is worth notice, "that the two latter are of thoſe ſpecies of ſeries which have hitherto "been deemed incapable of ſummation; the factors in the denomina-"tors of their terms being neither immediately conſequent to each "other, nor belonging to the ſame arithmetical progreſſion." In which remark the reader will obſerve ſomething indeed worth notice! The firſt ſeries (A) is ſummed by Mr. De Moivre, and it is very obvious, that $A-\frac{1}{6}$ is $=4D$; yet the ſeries D, our commentator tells us, is one of thoſe that have hitherto been deemed incapable of ſummation!

mation!—as is the ſeries C, though 3C is manifeſtly =4B, and the ſeries B is not reckoned in the ſame predicament with the ſeriés C and D!

At pa. 71 and 94 he introduces other *puffs!* by telling us, that two ſeries ſummed by Mr. LORGNA in his 4th ſection, and two ſeries ſummed by him in his 6th ſection, are of that order of ſeries which before were deemed incapable of being ſummed *algebraically*; though they *all* may readily be *ſo* ſummed by methods publiſhed many years before!

By the general tenor of his boaſting, I believe, moſt of his readers may have been led to imagine, that, in the book he has tranſlated, the ſums of many ſeries are aſſigned in finite *algebraic terms* which could not be *ſo* aſſigned by any method publiſhed before: but how will they be ſurpriſed at being aſſured, there is not *one* ſeries (whether with ſigns all poſitive, or alternately poſitive and negative) ſummed in *ſuch* terms in that book, which may not be ſummed *in the ſame terms*, not only by Mr. SIMPSON's method, but even by Mr. STIRLING's, exhibited in his *Methodus Differentialis*, publiſhed more than *half a century* ago!—It is ſurely very ſtrange, that amidſt all our commentator's boaſting, there ſhould not be found *one ſingle* ſeries ſummed *algebraically*, which may not be *ſo* ſummed by the method laſt mentioned! Yet ſo it is!—Nor is it leſs remarkable, that his author's boaſted *criterion of algebraic ſummation* (as it appears in his book) is no other than that which is plainly indicated by Mr. STIRLING's method, as well as by Mr. SIMPSON's.

In each ſeries (whoſe ſigns are all poſitive) ſummed algebraically in Mr. LORGNA's book, the factors in the denominators of the terms (agreeable to the known rule which indicates the poſſibility of *algebraic* ſummation) are either in the ſame arithmetical progreſſion, or may be eaſily reduced to the ſame. How ſhamefully then have the readers of that book and its tranſlation been amuſed and trifled with, by being told, that ſome of thoſe ſeries were before deemed incapable of being ſo ſummed, only becauſe their terms are diſguiſed by being drawn into certain general factors! Can the multiplying by a general factor, or two, render that ſeries incapable of algebraic ſummation which before was capable of ſuch ſummation? Surely it cannot.—The ſeries $\frac{1}{1.3.4}+\frac{4}{2.4.5}+\frac{7}{3.5.6}$ &c. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}$ &c. &c. for inſtance, appear (by the rule abovementioned) to be capable of algebraic ſummation; and ſo they are:—Is it not then a device, for the purpoſe of making the boaſted method appear to excel every other, that the ſeries $\frac{1}{3.6.28}+\frac{4}{6.8.35}+\frac{7}{9.10.42}$ &c. $\frac{1}{1.2.6}+\frac{3}{2.3.8}+\frac{5}{3.4.10}$ &c. &c. which are reſpectively equal to $\frac{1}{3.2.7}\times\overline{\frac{1}{1.3.4}+\frac{4}{2.4.5}+\frac{7}{3.5.6}}$ &c. $\frac{1}{2}\times\overline{\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}}$ &c. &c. are ſaid to have been deemed incapable of ſuch ſummation?

And is it not another device, for the ſame purpoſe, that we are told by the commentator (at pa. 40) " the ſummation of the ſeries " $\frac{1}{1.4}-\frac{1}{2.5}+\frac{1}{3.6}-\frac{1}{4.7}$ &c. is ſaid to be impoſſible by Mr. Stirling " and ſeveral other authors?"—I do not find, that Mr. Stirling has any where ſaid ſo!—That ſeries may certainly be ſummed by his method, though not in finite terms: and whether it may be ſummed by means of logarithms, or not, he has not ſaid.—Mr. Clarke has not ſhewn, that it may be ſummed *algebraically*.

Is it not obvious, that $\frac{1}{1.4}-\frac{1}{2.5}+\frac{1}{3.6}-\frac{1}{4.7}$ &c.

$$\text{is} = \tfrac{1}{3}\times\left\{\begin{array}{l} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\ \text{\&c.} \\ -\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}\ \text{\&c.} \end{array}\right.$$

$$= \tfrac{1}{3}\times\left\{\begin{array}{l} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\ \text{\&c.} \\ +1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\ \text{\&c.} \\ -1 + \frac{1}{2} - \frac{1}{3}\ ? \end{array}\right.$$

And, every novice knowing, that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ &c. is = Hyp. Log. of 2; who does not know, that the laſt expreſſion for the value of the ſeries in queſtion is $= \frac{2}{3}$ Hyp. Log. of $2 - \frac{5}{18}$?

The commentator, to magnify the importance of the buſineſs of ſumming ſeries whoſe terms have but one factor more in the denominator than in the numerator, fallaciouſly refers us, in his *preface*, to Mr. Simpson's and Mr. Stirling's writings; wiſhing us to believe, they looked upon that buſineſs as impoſſible to be performed by their methods.

With reſpect to Mr. Simpson, I have remarked in my *Obſervations*, pa. 21, that the reference is quite falſe!—and, with reſpect to Mr. Stirling, it may be obſerved, that, when he ſays (in his 2d *Prop. Summat. Serier.*) "Series ſummabitur, dummodo *a*, *b*, *c*, *d*, &c. ſint "integri, et numerus factorum in denominatore, ad minimum binario "excedat altiſſimum ipſius *z* dimenſionem in numeratore;" he is only ſpeaking of ſeries wherein the ſigns are all poſitive, and $x = 1$; which is the ſort of ſeries conſidered in that propoſition, and what he there ſays is undoubtedly true.—In his 3d Prop. he gives us a theorem by which we may readily find the ſums of that very ſort of ſeries, about the ſummation of which, the commentator makes ſuch extraordinary boaſt.

To do juſtice to Mr. Stirling, I think it right to inſert thoſe two propoſitions here; and to ſhew by examples, how all the ſeries, which Mr. Clarke ſays were deemed incapable of being ſummed *algebraically*, may be *ſo* ſummed by the *Differential Method*, publiſhed ſo long ago as the year 1730!

PROPOSITIO II.

"*Si termini cujuſcunque ſeriei formentur ſcribendo numeros quoſvis uni-* "*tate differentes in quantitate*

"$(T=)\ \frac{A}{z.z+1} + \frac{B}{z.z+1.z+2} + \frac{C}{z.z+1.z+2.z+3} + \frac{D}{z.z+1.z+2.z+3.z+4}$ &c.

" *Summa omnium terminorum in infinitum, incipiens ad datum quemlibet*
" *terminum, erit*

" $(S=)\ \frac{A}{z}+\frac{B}{2z.z+1}+\frac{C}{3z.z+1.z+2}+\frac{D}{4z.z+1.z+2.z+3}$ &c.

Ex. I. Taking the ſeries $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}$ &c. $(=S)$ from pa. 2 of the commentator's book; T will be $=\frac{1}{12z.z+1}$; and $S=\frac{1}{12z}=\frac{1}{12}$, when (to comprehend the whole ſeries) z is taken $=1$.

Ex. II. Taking $\frac{1}{8.18}+\frac{1}{10.21}+\frac{1}{12.24}$ &c. $(=S)$ the next ſeries in the ſame book; T will be $=\frac{1}{6z.z+2}=\frac{1}{6z.z+1}-\frac{1}{6z.z+1.z+2}$ (the values of z being 4, 5, 6, &c.); and S will be $=\frac{1}{6z}-\frac{1}{12z.z+1}$ $=\frac{2z+1}{12z.z+1}=\frac{3}{80}$, when (to comprehend the whole ſeries) z is taken $=4$.

Ex. III. Taking from pa. 46 of the ſame book, the ſeries $\frac{1}{2.6.8}+\frac{1}{3.8.10}+\frac{1}{4.10.12}$ &c. $(=S)$ which is adverted to at pa. 8 of this *Suppl.* we have $T=\frac{1}{4z.z+1.z+2}$ (the values of z being 2, 3, 4, &c.) and S will be $=\frac{1}{8z.z+1}=\frac{1}{48}$, when (to include the whole ſeries) z is taken $=2$.

Ex. IV. The ſeries $\frac{1}{2.6.4.5}+\frac{1}{4.8.5.6}+\frac{1}{6.10.6.7}$ &c. $(=S)$ being taken from pa. 66 of the ſame book; T will be $=\frac{1}{4z.z+2.z+3.z+4}$ $=\frac{1}{4z.z+1.z+2.z+3}-\frac{3}{4z.z+1.z+2.z+3.z+4}$; and S will be

=

$$= \frac{1}{12z.z+1.z+2} - \frac{3}{16z.z+1.z+2.z+3} = \frac{1}{48} \times \frac{4z+3}{z.z+1.z+2.z+3}$$

$= \frac{7}{2^7\,3^2}$, when z is taken $= 1$.

Ex. V. Taking, from pa. 84 of the ſame book, the ſeries $\frac{1}{3.5.7} + \frac{2}{4.6.8} + \frac{3}{5.7.9}$ &c. ($=S$) concerning which, Mr. CLARKE has given us (pa. 88, 89, 90, 91, 92, 93, 94) ſuch a ſtriking inſtance of his *pedantry*; we have $T = \frac{z-2}{z.z+2.z+4} = \frac{1}{z\,z+1} - \frac{7}{z.z+1.z+2} + \frac{18}{z.z+1.z+2.z+3} - \frac{18}{z.z+1.z+2.z+3.z+4}$ (the values of z being 3, 4, 5, &c.); and S will be $= \frac{1}{z} - \frac{7}{2z.z+1} + \frac{6}{z.z+1.z+2} - \frac{9}{2z.z+1.z+2.z+3} = \frac{2z^3+5z^2-z-3}{2z.z+1.z+2.z+3} = \frac{31}{240}$, when z is taken $=3$.

Ex. VI. Taking the ſeries $\frac{1}{1.2.6} + \frac{3}{2.3.8} + \frac{5}{3.4.10}$ &c. ($=S$) from pa. 85 of the ſame book; we have $T = \frac{2z-1}{2z.z+1.z+2} = \frac{1}{z.z+1} - \frac{5}{2z.z+1.z+2}$; and S will be $= \frac{1}{z} - \frac{5}{4z.z+1} = \frac{4z-1}{4z.z+1} = \frac{3}{8}$, when z is taken $= 1$.

"PROPOSITIO III.

" *Si termini cujuſvis ſeriei formentur ſcribendo numeros quoslibet unitate*
" *differentes pro* z *in quantitate*

$$\text{" (T)} = x^{z+n} \text{ in } \frac{a}{z} + \frac{b}{z.z+1} + \frac{c}{z.z+1.z+2} + \frac{d}{z.z+1.z+2.z+3} \text{ \&c.}$$

" *erit ſumma* (S) *æqualis* x^{z+n} *in*

$$\frac{a}{1-x.z}+\frac{b-Ax}{1-x.z.z+1}+\frac{c-2Bx}{1-x\,z.z+1.z+2}+\frac{d-3Cx}{1-x.z.z+1.z+2.z+3}\ \&c.$$

“ Quantitates A, B, C, D, &c. deſignant coefficientes terminorum “ præcedentes eos in quibus reperiuntur; ſcilicet eſt A $= \frac{a}{1-x}$, “ B $= \frac{b-Ax}{1-x}$, C $= \frac{c-2Bx}{1-x}$, &c. et ſic porro.”

A few examples of the uſe of this theorem will ſhew what ſort of foundation there was for our commentator’s extraordinary boaſts, and how well his references are to be relied on.

Ex. I. If we take the ſeries $\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{2.12}$ &c. (= S) from pa. 22 of his book; we ſhall have $T = \overline{-1}|^{z-1} \times \frac{1}{6z.z+2} = \overline{-1}|^{z-1} \times \overline{\frac{1}{6z.z+1}-\frac{1}{6z.z+1.z+2}}$ (x being $= -1$): and S will be $= \overline{-1}|^{z-1} \times \frac{1}{12z.z+1} = \frac{1}{24}$, when z is taken $= 1$; B being $= \frac{1}{12}$, and A, C, D, &c. each $= 0$.

Ex. II. Taking the ſeries $\frac{1}{1.9.10}-\frac{1}{2.12.12}+\frac{1}{3.15.14}-$ &c. (=S) from pa. 46 of the ſame book; we have $T = \overline{-1}|^{z-1} \times \frac{1}{6z.z+2.z+4}$ $= \overline{-1}|^{z-1} \times \overline{\frac{1}{6z.z+1.z+2}-\frac{1}{2z.z+1.z+2.z+3}+\frac{1}{2z.z+1\,z+2.z+3.z+4}}$ (x being $= -1$); and S will be $= \overline{-1}|^{z-1} \times \overline{\frac{1}{12z.z+1.z+2}-\frac{1}{8z.z+1.z+2.z+3}} = \overline{-1}|^{z-1} \times \frac{2z+3}{24z.z+1.z+2.z+3} = \frac{5}{3^2.8}$, when z is taken $= 1$; C being $= \frac{1}{12}$, D $= -\frac{1}{8}$, and A; B, E F, &c. each $= 0$.

Ex. III. The ſeries $\frac{1}{1.6.10.21} - \frac{1}{2.8.12.24} + \frac{1}{3.10.14.27} -$ &c. (= S) from pa 66 of the ſame book being propoſed; T will be =

$$\overline{-1}|^{z-1} \times \frac{1}{12z.z+2.z+4.z+6} = \overline{-1}|^{z-1} \times \overline{\frac{1}{12.z.z+1\ldots z+3} - \frac{1}{2z.z+1\ldots z+4} + \frac{5}{4z.z+1\ldots z+5} - \frac{5}{4z.z+1\ldots z+6}}$$

(x being $= -1$); and S will be

$$= \overline{-1}|^{z-1} \times \overline{\frac{1}{24z.z+1\ldots z+3} - \frac{1}{6z.z+1\ldots z+4} + \frac{5}{24z.z+1\ldots z+5}}$$

$$= \overline{-1}|^{z-1} \times \frac{z^2+5z+5}{24z.z+1\ldots z+5} = \frac{11}{24.720},$$

when z is taken $=1$; D being $= \frac{1}{48}$, E $= -\frac{1}{12}$, F $= \frac{5}{48}$, and A, B, C, G, H, &c. each $= 0$.

Ex. IV. Taking the ſeries $\frac{2}{1.3.3} + \frac{3}{3.5.9} + \frac{4}{5.7.27}$ &c. (= S) from pa. 69 of the ſame book; we have $T = \overline{\tfrac{1}{3}}|^{z+\frac{1}{2}} \times \frac{2z+3}{8z.z+1} =$

$$= \overline{\tfrac{1}{3}}|^{z+\frac{1}{2}} \times \overline{\frac{1}{4z} + \frac{1}{8z.z+1}}$$

(x being $= \frac{1}{3}$; and the values of z being $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, &c.) Conſequently S will be $= \overline{\tfrac{1}{3}}|^{z+\frac{1}{2}} \times \frac{3}{8z} = \frac{1}{4}$, when z is taken $= \frac{1}{2}$; A being $= \frac{3}{8}$, and B, C, &c. each $= 0$.

Ex. V. The ſeries $\frac{14}{4.8.2} - \frac{17}{6.10.4} + \frac{20}{8.12.8} -$ &c. (= S) being taken from pa. 70 of the ſame book; T will be $= \overline{-\tfrac{1}{2}}|^{z-2} \times \frac{3z+8}{8z.z+2}$

$$= \overline{-\tfrac{1}{2}}|^{z-2} \times \overline{\frac{3}{8z} + \frac{1}{4z.z+1} - \frac{1}{4z.z+1.z+2}}$$

(x being $= -\frac{1}{2}$; and the values of z being 2, 3, 4, &c.) Conſequently S will be $= \overline{-\tfrac{1}{2}}|^{z-2} \times \overline{\frac{1}{4z} + \frac{1}{4z.z+1}}$

$$= \overline{-\tfrac{1}{2}}|^{z-2} \times \frac{z+2}{4z.z+1} = \frac{1}{6},$$

when z is taken $= 2$; A being $\frac{1}{4}$, B $= \frac{1}{4}$, and C, D, &c. each $= 0$.

Ex. VI. Taking the ſeries $\frac{6}{2.8}-\frac{10}{4.12}+\frac{14}{6.16}-\frac{18}{8.20}$ &c. (= S) from pa. 73 of the ſame book; we have $T=\overline{-1}|^{z-1}\times\frac{2z+1}{4z.z+1}=\overline{-1}|^{z-1}\times\overline{\frac{1}{2z}-\frac{1}{4z.z+1}}$ (x being $=-1$); and S will be $=\overline{-1}|^{z-1}\times\frac{1}{4z}=\frac{1}{4}$, when z is taken $=1$; B, C, &c. being each $=0$.

Ex. VII. The ſeries $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}$ &c. (= S) being taken from the ſame page; T will be $=\overline{-1}|^{z-\frac{1}{2}}\times\frac{2z+1}{8z.z+1}=\overline{-1}|^{z-\frac{1}{2}}\times\overline{\frac{1}{4z}-\frac{1}{8z.z+1}}$ (x being $=-1$; and the values of z being $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c.) Conſequently S will be $=\overline{-1}|^{z-\frac{1}{2}}\times\frac{1}{8z}=\frac{1}{12}$, when z is taken $=\frac{3}{2}$; B, C, &c. being each $=0$.

Ex. VIII. Taking the ſeries $\frac{2}{1.3.5}-\frac{3}{2.4.6}+\frac{4}{3.5.7}-$ &c. $(=S)$ from pa. 84 of the ſame book; T will be $=\overline{-1}|^{z-1}\times\frac{z+1}{z.z+2.z+4}=$ $\overline{-1}|^{z-1}\times\overline{\frac{1}{z.z+1}-\frac{4}{z.z+1.z+2}+\frac{9}{z.z+1.....z+3}-\frac{9}{z.z+1.....z+4}}$ (x being $=-1$): and S will be $=\overline{-1}|^{z-1}\times\overline{\frac{1}{2z.z+1}-\frac{3}{2z.z+1.z+2}+\frac{9}{4z.z+1.z+2.z+3}}$ $=\overline{-1}|^{z-1}\times\frac{2z^2+4z+3}{4z.z+1.z+2.z+3}=\frac{3}{32}$, when z is taken $=1$; A, E, F, &c. being each $=0$.

Ex. IX. If we take the ſeries $\frac{5.7}{1.2.3}-\frac{8.9}{2.3.4}+\frac{11.11}{3.4.5}-\frac{14.13}{4.5.6}$ &c. $(=S)$ which is mentioned pa. 5 of this *Suppl.* and may be found pa. 12 of Mr. Clarke's *Addit. Rem.* T will be $=\overline{-1}|^{z-1}\times\frac{3z+2.2z+5}{z.z+1.z+2}=\overline{-1}|^{z-1}\times$

$\overline{\frac{6}{z} - \frac{1}{z.z+1} - \frac{4}{z.z+1.z+2}}$ (x being $=-1$): and S will be $=$ $\overline{-1}|^{z-1} \times \frac{3}{z} + \frac{2}{z.z+1} = \overline{-1}|^{z-1} \times \frac{3z+5}{z.z+1} = 4$, when z is taken $=1$, C, D, &c. being each $= o$.

The value of T being $\frac{N}{z.z+a.z+b(n)}$, where N is ſuppoſed $= Az^{n-2} + Bz^{n-3}$ &c. if a, b, &c. be any poſitive integers, the value of S will be aſſignable in *algebraic* terms, by Prop. II. the ſigns in the ſeries (S) being all poſitive.

If, the terms being the ſame, the ſigns be alternately poſitive and negative, and the ſeries be then denoted by V; the ſeries ($\overset{\prime}{S}$) of the poſitive terms only will be

$$\frac{N}{z.z+a.z+b(n)} + \frac{\overset{\prime\prime\prime}{N}}{z+2.z+a+2.z+b+2(n)} \text{ \&c.}$$

$$= \frac{N}{2^{n}.y.y+\frac{1}{2}a.y+\frac{1}{2}b\,(n)} + \frac{\overset{\prime\prime\prime}{N}}{2^{n}.y+1.y+\frac{1}{2}a+1.y+\frac{1}{2}b+1\,(n)} \text{ \&c.}$$

y being $= \frac{1}{2}z$.

Therefore, if a, b, &c. be *even* poſitive numbers, the value of $\overset{\prime\prime}{S}$ will alſo be aſſignable in *algebraic* terms by Prop. II. and conſequently the value of V ($=2\overset{\prime\prime}{S}-S$) will then be aſſignable in *ſuch* terms by that propoſition.

This is the very ſame rule, or *criterion*, as Mr. LORGNA has given, reſpecting the algebraic ſummation, about which ſo much boaſt is made in Mr. CLARKE's book! this having the advantage of being general for any number of factors, whereas Mr. LORGNA's relates only to the few particular caſes which he has conſidered.

But the ſummation may be *algebraic* in other caſes, as well when the ſigns are alternately poſitive and negative, as when they are all poſitive;

tive; as I have ſhewn by Mr. SIMPSON's method: and the ſame appears by Mr. STIRLING's method; of which the following are inſtances.

Ex. I. S being $= \frac{1}{r.r+1.r-\frac{1}{2}.r+\frac{3}{2}} + \frac{1}{r+1.r+2.r+\frac{1}{2}.r+\frac{5}{2}}$ &c.

$T = \frac{1}{z.z+1.z-\frac{1}{2}.z+\frac{3}{2}}$ will be $= \frac{4}{3.z-\frac{1}{2}.z+\frac{3}{2}} - \frac{4}{3z.z+1} = \frac{4}{3y.y+2} - \frac{4}{3z.z+1} = \frac{4}{3y.y+1} - \frac{4}{3y.y+1.y+2} - \frac{4}{3z.z+1}$, y being put for $z-\frac{1}{2}$:

and S, by Prop. II. is found $= \frac{4}{3y} - \frac{2}{3y.y+1} - \frac{4}{3z} = \frac{1}{3} \times \frac{1}{z-\frac{1}{2}.z.z+\frac{1}{2}}$.

Ex. II. S being $= \frac{\overline{r+\frac{1}{2}}|^2}{r.r+1.r-\frac{1}{2}.r+\frac{3}{2}} + \frac{\overline{r+\frac{3}{2}}|^2}{r+1.r+2.r+\frac{1}{2}.r+\frac{5}{2}}$ &c.

$T = \frac{\overline{z+\frac{1}{2}}|^2}{z.z+1.z-\frac{1}{2}.z+\frac{3}{2}}$ will be $= \frac{4}{3.z-\frac{1}{2}.z+\frac{3}{2}} - \frac{1}{3z.z+1} = \frac{4}{3y.y+2} - \frac{1}{3z.z+1} = \frac{4}{3y.y+1} - \frac{4}{3y.y+1.y+2} - \frac{1}{3z.z+1}$, y being put for $z-\frac{1}{2}$:

and S, by Prop. II. will be $= \frac{4}{3y} - \frac{2}{3y.y+1} - \frac{1}{3z} = \frac{3z^2+\frac{1}{4}}{3.z-\frac{1}{2}.z.z+\frac{1}{2}}$.

It is obſervable, that the ſeries here exceeds the ſeries in the preceding example by the quantity $\frac{1}{z}$.

Ex. III. S being $= \frac{1}{r.r+1.r+3.r+4} - \frac{1}{r+1.r+2.r+4.r+5} +$ &c.

T will be $= \overline{-1}|^{z-r} \times \frac{1}{z.z+1.z+3.z+4} =$

$\overline{-1}|^{z-r} \times \overline{\frac{1}{z.z+1.z+2.z+3} - \frac{2}{z.z+1.z+2.z+3.z+4}}$: and S, by Prop. III. will be $= \overline{-1}|^{z-r} \times \frac{1}{2z.z+1.z+2.z+3}$; D being $= \frac{1}{2}$, and A, B, C, E, F, &c. each $= 0$.

It is obvious that, by this method of Mr. STIRLING, the ſum of any propoſed finite number of terms, in any ſuch ſeries as the foregoing ones, may be very readily found by taking z of ſuch a value in the expreſſion for the value of S, that ſuch expreſſion ſhall give the ſum of the remaining terms of the ſeries when that finite number (ſuppoſed at the beginning of the ſeries) ſhall be taken from it: for then, ſuch ſecond value of S being taken from the firſt value thereof, the difference will be the ſum ſought.

In the eaſy manner above explained, are ſeries (whether with ſigns all poſitive, or alternately poſitive and negative) ſummed *algebraically* by Mr. Stirling's theorems; even *all* thoſe which we are told, in Mr. Clarke's book, were deemed incapable of *algebraic* ſummation!

Did he underſtand Mr. Stirling's and Mr. Simpson's writings, or did he not?—If he did,—how groſly has he impoſed upon his readers by falſe references, aſſertions, and inſinuations!—If he did not,—how vain was the man to ſet himſelf up for a *Commentator!*

He has ſaid (pa. 4 of his *Addit. Rem.*) that Mr. Simpson did not know the uſe of his own theorems! Will he ſay ſo of Mr. Stirling (becauſe the latter, like the former, has not been diffuſe in examples)? and tell the world, that neither he, nor any other perſon, ever thought of applying the above propoſitions as I have done?

There is an elegance in Mr. Stirling's propoſitions which pleaſes me much: and I believe I may venture to aſſert, that every ſeries of the claſs we are conſidering (whether with ſigns all poſitive, or alternately poſitive and negative) which can be ſummed *algebraically*, may (without any idea of *fluxions*) be *ſo* ſummed by his method.—When the ſeries propoſed is incapable of *algebraic* ſummation, his method (it is well known) tranſmutes it into another ſeries converging more ſwiftly than the original one: whereas the ſum, by other methods, may then be aſſigned by means of circular arcs and logarithms;—but by none other ſo readily as by Mr. Simpson's method.—Which, as well as Mr. Stirling's, is an *original* method; wherein the manner in which the ſeries may be produced (by repeatedly taking fluents and fluxions) is not conſidered, as it is in Mr. De Moivre's method; which is alſo an *original* one. It is after this laſt mentioned method (which Mr. De Moivre had not carried to any conſiderable length) that Mr. Lorgna has proceeded; and carried it to a greater length, by means of the equation $fl.\,v\dot{x} = vx - fl.\,\dot{v}x$ (made uſe of by Mr. De Moivre): but he has not given a general theorem, like Mr. Simpson's; nor is his manner of evolving the value of the fluential ariſing in the proceſs ſuch as any analyſt of judgment would chuſe to purſue, who may have obſerved how much more eaſily and perſpicuouſly the deſired concluſion may be obtained by Mr. Simpson's method.

We find no boaſts in Mr. Stirling's and Mr. Simpson's books, about the *algebraic* ſummation of ſeries, though they have given general theorems for any number of factors; whilſt Mr. Lorgna has only given a few particular theorems, wherein the number of factors in the denominators of the terms of the ſeries does not exceed *four*.

The world, I verily believe, cannot produce another mathematical book ſo ſtuffed with boaſts, and miſrepreſentations, as Mr. Clarke's is!

And his remarks upon my *Obſervations* are not leſs extravagant than the paſſages laſt alluded to:—ſurely, for perverſe cavil and rancorous railing, nothing that ever was called reaſoning can equal them!——Sometimes the man's reaſon ſeems to be entirely loſt in furious paſſion, that vents itſelf in the moſt illiberal reflections and ſpiteful ſuggeſtions! which indeed, being of no weight in the balance of truth, are to be regarded only as bubbles blown up by the breath of malice.

But all this fury may be eaſily accounted for.—Enough, it ſeems, has happened to the poor man, to diſturb his mind, and put him out of temper!—He has been a candidate for an honour, of which he was not thought worthy! and in the rage of diſappointment he attributes his want of ſucceſs to my adverſe influence!—whereas, the truth is, I even did not know any thing of the attempt alluded to, until the fatal evening was paſt that blaſted his hopes.

It is a plain proof that his reaſon is not perfectly right, to ſuppoſe that I could wiſh to have been named with others (in his Preface) merely to be degraded (as thoſe others ſhamefully are) by a compariſon with Mr. Lorgna, who is (there) repreſented as ſurpaſſing them all very greatly:—with what truth, my *Obſervations* perhaps may aſſiſt the reader in judging.

The remarker takes care to let us know, that his author is a great man, a gentleman and a ſcholar; and to give us broad hints, that he thinks himſelf of ſome conſequence!—But what is all that, to the determining the merits, or demerits of the work in queſtion?—A man may be reputed a gentleman and a ſcholar, and at the ſame time not know much of the doctrine of ſeries.—For my part, I am ſo little diſpoſed to trouble the public with any account of myſelf, that I ſhall only in one reſpect ſay, not what I am, but what I am not:—I am not ſo indifferent in the cauſe of Truth and Science, as to forbear cenſuring a book, when I find it contains only *pretended* improvements inſtead of *real* ones; even though the writer of it ſhould happen to be a finer gentleman, and a greater ſcholar than the author, or commentator, ſo often alluded to.

I ſhall only make a few more obſervations on our enraged commentator's frothy remarks, and then leave him to his own meditations.

The transformation of the general theorem, ſo as to diſcover the *criterion* of algebraic ſummation, being performed by Mr. Simpson, in a peculiar manner, in his 4th propoſition; (that is, by taking as many terms of his ſeries S as may be requiſite, and comparing the remaining terms with the ſeries $\frac{1}{p} \pm \frac{z^n}{p+n}$ &c. $\frac{1}{q} \pm \frac{z^n}{q+n}$ &c. &c.); the commentator is reduced to this *dilemma*, either to confeſs that my theorems (alluded to in his 1ſt Object. in his *Suppl.*) may be deduced from what Mr. Simpson has done in his *Diſſertations*, without adopting the ſame method of procedure as laid down by Mr. Lorgna, or that this laſt mentioned gentleman has followed Mr. Simpson in his method of procedure!

And I have, in my *Obſervations*, not only ſhewn Mr. LORGNA's *criterion*, or *indubitable mark* of algebraic ſummation, to be defective in two inſtances; but, in my *final remark* there, I have given a rule, or rules, for finding many more ſuch inſtances: and in my *Appendix*, I have farther pointed out its defects; and have proved, that there are innumerable caſes wherein ſuch ſummation is poſſible which will not be diſcovered by his imperfect (pretended indubitable) mark.

Yet, notwithſtanding the proofs are ſo plain, the cavilling remarker (in his *Addit. Rem.* pa. 4, 5,) determined to contradict every thing I advance (wrong or right, no matter which!) has attempted to evade the force of thoſe concluſions, in his uſual manner, by malapert declamation, without even a ſhadow of argument!—And, in the ſame manner, he runs on, at pa. 7, in contradiction to what I have ſaid of his note, pa. 45 of his *Supplement*, which I have aſſerted is unneceſſary: and ſo it certainly is. For p may be always ſuppoſed the leaſt of the quantities p, q, r, &c. and being ſo, when n is equal to, or greater than p, m muſt be $=p$.—In other caſes m may be taken at pleaſure, under the reſtrictions mentioned pa. 6 of my *Appendix*.—The remarker frivolouſly contends, that q, or r, &c. may (as well as p) be ſuppoſed the leaſt of the quantities p, q, r, &c. So it may be: but the ſuppoſing p to be the leaſt is ſufficient, and therefore any other ſuppoſition is unneceſſary.

When I had ſhewn ſome of Mr. LORGNA's theorems to be ill expreſſed and defective, his commentator, inſtead of abiding by the aſſertion, that nothing more could be expected, or wiſhed for in the doctrine of ſeries, ſet himſelf to write a *Supplement*, and to make ſome improvements upon what his author had done!—and we have ſeen what wonderful improvements he has made!—After writing a great deal at pa. 5, 6, (as his author had done before, pa. 97, 98, 99) about ſumming ſuch ſeries as

$$\frac{4.31}{8.8.23} - \frac{7.41}{13.14.28} + \frac{10.51}{18.20.33} - \text{\&c.}$$

$$\frac{5.25}{1{,}8.10} - \frac{7.35}{2.10.14} + \frac{9.45}{3.12.18} - \&c.$$

$$\frac{10.12}{7.13.24} - \frac{12.19}{9.15.38} + \frac{14.26}{11.17.52} - \&c.$$

as examples of ſeries whoſe terms conſiſt of *two* factors in each numerator, and *three* factors in each denominator; he leaves us to make what we can of his *futile* remarks, with the ſatisfaction of knowing, that ſuch ſeries are nothing at all to the purpoſe; being no other than

$$\frac{1}{2} \times \overline{\frac{31}{8.23} - \frac{41}{13.28} + \frac{51}{18.33} - \&c.}$$

$$\frac{5}{4} \times \overline{\frac{5}{1.4} - \frac{7}{2.5} + \frac{9}{3.6} - \&c.}$$

$$\frac{5}{7.13} - \frac{6}{9.15} + \frac{7}{11.17} - \&c.$$

which, having but *one* factor in each numerator, and *two* factors in each denominator, relate to a preceding ſection!

And, preſently after, he gives us other ſhallow, inſignificant remarks (pa. 6, 7, of his *Suppl.)* as I have obſerved, pa. 14 of my *Appendix*—Upon thoſe laſt mentioned remarks, the remarker exclaims (pa. 14 of his *Addit. Rem.)* that I have endeavoured to caſt an odium. —It is true, I have cenſured them; and they deſerve cenſure, being not only *ſhallow* and *inſignificant* in one reſpect, but even in another reſpect abſolutely *falſe!*—as are his remarks, pa. 3 of his *Suppl.*—His additional remarks (pa. 15, 16) by which he meant to explain thoſe other, likewiſe deſerve the ſame cenſure!

If his firſt ſuppoſition, $\frac{P}{\omega}+\frac{Q}{\omega-\pi}=0$, be true; his ſecond ſuppoſition, $\frac{PX^{\omega}}{\omega}+\frac{QX^{\omega-\pi}}{\omega-\pi}=0$, can only be ſo when X^{π} is $=1$; and all his reaſoning reſpecting other values of X is falſe!

The requiſite conditions are not, that

π and $\omega - \delta$ be poſitive integers,

$$\text{and } \frac{PX^{\omega}}{\omega} + \frac{QX^{\omega-\pi}}{\omega-\pi} + \frac{RX^{\omega-\delta}}{\omega-\delta} - \frac{R}{\omega-\delta} = o,$$

as he erroneouſly makes them: but that

π and $\omega - \delta$ be poſitive integers,

and, at the ſame time,

$$\frac{PX^{\pi}}{\omega} + \frac{Q}{\omega-\pi} = o, \text{ and } \frac{RX^{\omega-\delta}}{\omega-\delta} - \frac{P}{\omega} - \frac{Q}{\omega-\pi} - \frac{R}{\omega-\delta} = o.$$

Surely then we may conclude, that he was *not a little puzzled*, when he wrote the remarks in his *Supplement*, and even whilſt he was attempting to explain them by his additional ones!—Ridiculous indeed, for a man to ſet himſelf up for a guide through intricacies, of which he has ſo little knowledge, as to be bewildered the moment he ſteps out of the common way!

A general and certain method of determining the caſes of *algebraic* ſummation is pointed out in the *Final Remark* in my *Obſervations*:—the equating the coefficients for that purpoſe (as mentioned in his *Addit. Rem.* pa. 16) cannot always be perfectly ſatisfactory, as that proceſs may ſometimes prove fallacious;—eſpecially if managed with ſo little ſkill as he has ſhewn in his puerile attempt juſt now expoſed.

The commentator, in his *Supplement*, having, with reſpect to the ſeries $\frac{ab(k-1)}{p\overset{''}{p}\overset{'''}{p}(k)} + \frac{\overline{a+1}.\overline{b+1}(k-1)\times x}{\overline{p+1}.\overline{\overset{''}{p}+1}.\overline{\overset{'''}{p}+1}(k)}$ &c. aſked, " what Author be-
" ſides Mr. Lorgna hath ever ſhewn, in a *general manner*, the impoſ-
" ſibility of the ſummation in the former caſe (that is when x is $= 1$);
" and the conditions or neceſſary relation of the factors for the alge-
" braic ſummation in the latter" (that is when x is $= -1$); I conſidered the two parts of the queſtion diſtinctly, as two queſtions. The doing ſo, the cavilling remarker calls *garbling!* And, though he ſeems to admit, that what I have ſaid to the firſt part is true; yet he carps, in his uſual manner, at my reply to the ſecond part: which reply runs thus,

thus, " unfortunately it does not appear, that Mr. Lorgna has given " any *general* theorem reſpecting the ſummation of the ſeries alluded " to, when x is $= -1$!" An aſſertion ſtrictly true;—and to which I ſhould have added,—*only the two caſes wherein* k *is not greater than 3 are conſidered by him*; *and even in one of thoſe caſes he has betrayed his ignorance of that very relation of the factors which his puzzled commentator ſpeaks of, as I have already ſhewn!*

The remarker, with regard to the ſeries laſt written, ſays, in his *Addit. Rem.* pa. 21, " nothing can be more obvious than, in Mr. " Lorgna's proceſs, there will *always* finally reſult, in the caſe of the " number of *factors* under conſideration, the term $\frac{1+x^n}{1\mp x}$" (rather $C \times fl. \frac{\overline{1+x^n}.x^m\dot{x}}{1\mp x}$) " which, when the ſigns are +, requires the *upper* " ſign in the denominator, and is therefore *infinite* when x is $=1$; but " when the ſigns are + and −, the *lower* ſign is uſed, and the expreſ- " ſion to which it appertains becomes an *algebraic* quantity when n is " an *odd* number."—Now it is obſervable, he takes no notice of the coefficient* of his term $\frac{1+x^n}{1\mp x}$; which, it was incumbent on him to have ſhewn, from what appears in Mr. Lorgna's book, can in no caſe be $= o$, when x is $=1$. It was alſo neceſſary, that he ſhould have ſhewn, why the ſummation is not always *algebraic* when the differences (of the factors in the denominators) are all *odd* numbers, and the ſigns + and − alternately; as, from what he ſays, the reader would be led to infer, that it always muſt be *ſo*, in that caſe.—And it would not have been amiſs, if he had ſhewn what will be the conſequence, with reſpect to ſuch ſummation, when thoſe differences are ſome of them *even* and ſome of them *odd* numbers.—But it ſeems, ſuch explanations are beyond our dabbler's depth!

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* It ſeems indeed, by what he ſays pa. 20, that he believes it will *always* be $=1$; but ſurely he cannot be *quite ſo ignorant!*

From Mr. Simpson's proceſs, by a pure algebraic inveſtigation, it appears, that, when b is leſs than $k-1$; and the values of $\overset{\prime}{A}$, $\overset{\prime\prime}{A}$, $\overset{\prime\prime\prime}{A}$, &c. are as expreſſed at pa. 22 of my *Appendix*;

$$\overset{\prime}{A} + \overset{\prime\prime}{A} + \overset{\prime\prime\prime}{A}\ (k) \text{ is always } = 0:$$

and that, when the values of $\overset{\prime}{A}$, $\overset{\prime\prime}{A}$, $\overset{\prime\prime\prime}{A}$, &c. are as expreſſed at pa. 20,

$$\overset{\prime}{A} + \overset{\prime\prime}{A} + \overset{\prime\prime\prime}{A}\ (k) \text{ is always } = 1.$$

Which equations (by means whereof the new theorems at pa. 20, 21, 22 of my *Appendix* are deduced) are perhaps the moſt uſeful ones we can obtain reſpecting the relation of the coefficients of the terms in the general value of the ſeries under conſideration. Are thoſe equations indicated by Mr. Lorgna's method? or can they be obtained thereby, otherwiſe than by a tedious, unſatisfactory induction?

If $\overset{\prime}{A}$ be $= \dfrac{a-p.b-p}{\overset{\prime\prime}{p}-p.\overset{\prime\prime\prime}{p}-p}$, $\overset{\prime\prime}{A} = \dfrac{a-\overset{\prime\prime}{p}.b-\overset{\prime\prime}{p}}{p-\overset{\prime\prime}{p}.\overset{\prime\prime\prime}{p}-\overset{\prime\prime}{p}}$, $\overset{\prime\prime\prime}{A} = \dfrac{a-\overset{\prime\prime\prime}{p}.b-\overset{\prime\prime\prime}{p}}{p-\overset{\prime\prime\prime}{p}.\overset{\prime\prime}{p}-\overset{\prime\prime\prime}{p}}$; and $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$ be any poſitive integers; the ſeries

$$Q = \frac{ab}{p\overset{\prime\prime}{p}\overset{\prime\prime\prime}{p}} + \frac{a+1.b+1.x}{p+1.\overset{\prime\prime}{p}+1.\overset{\prime\prime\prime}{p}+1} + \frac{a+2.b+2.x^2}{p+2.\overset{\prime\prime}{p}+2.\overset{\prime\prime\prime}{p}+2} \text{ \&c.}$$

$$\text{will be} = \frac{1}{x^p} \times \overline{\overset{\prime}{A} + \overset{\prime\prime}{A}x^{p-\overset{\prime\prime}{p}} + \overset{\prime\prime\prime}{A}x^{p-\overset{\prime\prime\prime}{p}}} \times \text{fl.}\,\frac{x^{p-1}\dot{x}}{1-x}$$

$$- \overset{\prime\prime}{A}x^{p-\overset{\prime\prime}{p}} \times \overline{\frac{1}{p} + \frac{x}{p+1} + \frac{x^2}{p+2}\ (\overset{\prime\prime}{p}-p)}$$

$$- \overset{\prime\prime\prime}{A}x^{p-\overset{\prime\prime\prime}{p}} \times \overline{\frac{1}{p} + \frac{x}{p+1} + \frac{x^2}{p+2}\ (\overset{\prime\prime\prime}{p}-p)}$$

Where, when x is $= 1$, $\overset{\prime}{A} + \overset{\prime\prime}{A}x^{p-\overset{\prime\prime}{p}} + \overset{\prime\prime\prime}{A}x^{p-\overset{\prime\prime\prime}{p}}$ will be $= 1$. Therefore, it is manifeſt, the ſame expreſſion will be likewiſe $= 1$, when $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$ being *even* numbers, x is $= -1$.

If, x being $= -1$, $\overset{\prime\prime}{p}-p$ be an *even* number, and $\overset{\prime\prime\prime}{p}-p$ an *odd* number, $x^p \times$ the coefficient of fl. $\dfrac{x^{p-1}\dot{x}}{1-x}$, in the value of Q, will be

$\overset{\prime}{A} + \overset{\prime\prime}{A}$

$\overset{\prime}{A}+\overset{\prime\prime}{A}-\overset{\prime\prime\prime}{A}=2\overset{\prime}{A}+2\overset{\prime\prime}{A}-1$, ſeeing that $\overset{\prime}{A}+\overset{\prime\prime}{A}+\overset{\prime\prime\prime}{A}$ is $=1$. Conſequently, that the coefficient of $fl.\frac{x^{p-1}\dot{x}}{1-x}$ may vaniſh, and the value of the ſeries Q be expreſſible in algebraic terms $2\overset{\prime}{A}+2\overset{\prime\prime}{A}-1$ muſt be $=o$, and $\overset{\prime}{A}+\overset{\prime\prime}{A}=\frac{1}{2}$; or, which comes to the ſame thing, $\overset{\prime\prime\prime}{A}=\frac{1}{2}$. Moreover, if $\overset{\prime\prime}{p}-p$, $\overset{\prime\prime\prime}{p}-p$ be both *odd* numbers, $x^{p}\times$ the coefficient of $fl.\frac{x^{p-1}\dot{x}}{1-x}$, when x is $=-1$, will be $=\overset{\prime}{A}-\overset{\prime\prime}{A}-\overset{\prime\prime\prime}{A}$; and conſequently, that it may vaniſh, $\overset{\prime}{A}$ muſt be $=\frac{1}{2}$; or, which comes to the ſame thing, $\overset{\prime\prime}{A}+\overset{\prime\prime\prime}{A}=\frac{1}{2}$. And the theorem, in theſe caſes, may ſtand as at pa. 20, 21 of my *Appendix*, when k is there taken $=3$: which, without any defect, will comprehend all the poſſible caſes wherein the value of Q may be expreſſed *algebraically*.

In this eaſy, conciſe manner, is the theorem inveſtigated by Mr. Simpson's method!—Is Mr. Clarke's method of inveſtigating the ſame theorem to be compared with this, for facility, perſpicuity, and elegance?—Surely it is not. He himſelf, if he has a ſpark of candour, muſt acknowledge how much ſuperior this method is to his! eſpecially when he counts the number of pages he has filled in his *Addit. Rem.* in deducing the ſame reſult.

With regard then to the obtaining theorems for the *algebraic* ſummation, when the relation of certain coefficients is to be conſidered, Mr. Simpson's method is manifeſtly much ſuperior to Mr. Lorgna's: and, I preſume, I have clearly proved, that, in no reſpect whatever, is the latter ſuperior to the former.

Thus have I ſhewn, that our pert, pedantic commentator, to magnify the aſſumed importance of his paltry production, has fallaciouſly *cried up* (in diſparagement of the works of others) a method as *intirely new* and *much more general* than any other on the ſame ſubject! has unjuſtly imputed *impropriety* and *narrowneſs* to Mr. Stirling's and Mr.

SIMPSON's methods! and has aſſerted other *arrant falſities* reſpecting the impoſſibility of *algebraic* ſummation by any method that had appeared prior to his author's!

Surely theſe are facts that ought to be expoſed, to do juſtice to the memory of the dead!—And the boaſter, I think, may be heartily aſhamed, that he has given occaſion for theſe animadverſions!

Let him then ceaſe his vaunting, and ſneak off in ſilence!—The mathematical world will now judge in what eſtimation he and his *catch-penny* publication ought to be held; and will cenſure, according to his demerits, the man who, by ſuch *notorious miſrepreſentations*, has rendered himſelf obnoxious to *public* DISGRACE!

END OF THE SUPPLEMENT.

ERRATUM.

Page 4, line 5, *for* 2, 3, 4, *read* 7, 8, 9.

NOTES.

HAVING lately ſeen Mr. Lorgna's ***original book*** on Series, and compared it with the Tranſlator's book, I noted therefrom the following particulars.

1. Mr. Lorgna's *Title* is—" Specimen de Seriebus Convergentibus." —Which is proper.

His Tranſlator calls the Work,—" A Diſſertation on the Summation " of Infinite Converging Series with Algebraic Diviſors: exhibiting a " Method, not only entirely new, but much more general than any " other which has hitherto appeared on the ſubject!"

Which is an ***empty boaſt***, and therefore ***highly improper!***

2. Mr. Lorgna concludes his *Preface* with theſe words,—" Siquid " Serierum doctrinæ exinde accedat incrementum, fineſque proferan- " tur, viderint æqui rerum æſtimatores, ſi quos hæc noſtra perlegere " non piguerit."—Which are not improper.

His Tranſlator ſays,—" I think we may ſafely conclude with our " Author, that nothing more can now be expected or wiſhed for in the " doctrine of Series; being brought to its greateſt perfection."

Which ſurely is ſuch ***abſurd bombaſt*** as a man, who really knows any thing of that doctrine, would be aſhamed to write!

3. Mr. Lorgna, ſpeaking of the ſeries $\frac{1}{1\cdot4}-\frac{1}{2\cdot5}+\frac{1}{3\cdot6}-\frac{1}{4\cdot7}$ &c. ſays of Mr. Stirling,—" quæ quo pacto ſummari poſſit ex præcla- " riſſimo auctore non apparet."

Which is true, if only meant to note the omiſſion of that ſeries as a particular example.

His Tranſlator tells us,—" The ſummation of this ſeries is ſaid to " be impoſſible by Stirling, and ſeveral other authors."—Which is ***not true.*** See pa. 10 of this *Suppl.*

4. Mr. Lorgna, treating of ſeries with four factors in each denominator (the numerators being each = 1) ſays,—quæ vero ſignis affici-
" untur alternantibus erunt Algebraice ſummabilis, ſi π, ω, δ fuerint
" integri pares."—Which is true.

His Tranſlator ſays,—" But if the ſigns be alternately + and –,
" the ſummation cannot be ſo exhibited, unleſs π, ω, and δ be *even*
" whole numbers.—Which is *not true*: for the ſummation may not only be always exhibited *algebraically* when π, ω, and δ are *even* whole numbers; but it may ſometimes be *ſo* exhibited when π, ω, and δ are not *even* whole numbers: as is ſhewn in my *Obſervat*. and *Append.*

5. Mr. Lorgna no where mentions Mr. Simpson; nor does it appear, that he had read his *Diſſertations*:—He therefore cannot be ſaid to have depreciated that work particularly.

His officious Tranſlator names Mr. Simpson expreſſly; miſrepreſents a paſſage in his book; and diſparages his method, by imputing *impropriety* and *narrowneſs* to it.

If the *Tranſlator* had followed his *Author* more faithfully, without foiſting a *new Title* to the work, and the name of Mr. Simpson into his preface; he would have rendered his book leſs objectionable, and himſelf leſs reprehenſible.

☞ It is truly remarkable, that about *one fourth* of the examples in both the *original* and *tranſlation* are *materially wrong!*—even thoſe are *ſo* (at pa. 66) which the tranſlator imagined he had corrected!

If the *tranſlation* be not worth reading for the inſtruction it can afford; ſome collectors of rarities may however think it worth preſerving as a *curioſity*, being perhaps the moſt extraordinary production of the kind for *puffs* and *blunders!*

POSTCRIPT.

POSTSCRIPT.

I HAVE, in the Poſtſcript to my *Appendix*, truly declared, that I had not ſeen certain works mentioned in Mr. Clarke's *Supplement*; and I now, with the ſame regard to truth, and the ſame confidence that I ſhall be believed by every man of character and reputation, make a farther declaration reſpecting my knowledge (or rather want of knowledge) of what may have been done by *foreign* mathematicians. —I have not read any (not even ſo much as *one)* of the *Acta Eruditorum*; or of the *foreign Diaries*; or of the *Hiſtories*, *Memoirs*, *Journals*, *Acts*, *Tranſactions*, or *Commentaries*, of any of the *foreign Academies*, or *Societies*:—But I would be underſtood to except ſuch papers, or eſſays, as may have been reprinted and inſerted in the few volumes which I have of the works of Prestet, Ozanam, Huygens, John and Daniel Bernoulli, L'Hospital, Bion, Fontenelle, Bougainville, Euler, and D'Alembert.

The Father of lies himſelf could not ſuggeſt a reflection more unjuſt than that in Mr. Clarke's *Additional Remarks* and *Poſtſcript*, reſpecting the information he wiſhes the public to believe I have derived from the writings of *foreign authors!*—The man who can ſo baſely attempt to detract from the reputation of another, out of *mere ſpite*, without *any ground* for the opinion he endeavours to propagate, muſt be *vile indeed!* and *ſhame* will be his portion!

Ignorance might have been pleaded in excuſe for the miſrepreſentations which I have expoſed in his work, if his exceſſive vanity had not prompted him to wave that plea, and truſt to his effrontery:—but no plea can be offered in excuſe for the *baſeneſs of diſpoſition* manifeſted in thus aiming ardently to propagate the groundleſs inſinuations juſt now alluded to;—he here ſtands, without excuſe, an object of *cenſure* and *contempt!*

His impertinent, idle conceit, reſpecting a meeting in a future ſtage of exiſtence, renders him, if poſſible, ſtill more contemptible!—I might predict, perhaps from principles not without foundation, that real heroes in mathematical literature (even thoſe whoſe works I may have occaſionally cenſured) will there take me by the hand, as a *genuine* friend to *Truth* and their favourite *Science*; whilſt they turn their backs on him, as a *baſe counterfeit* unworthy of their regard.

FINIS.

Fig. 1.

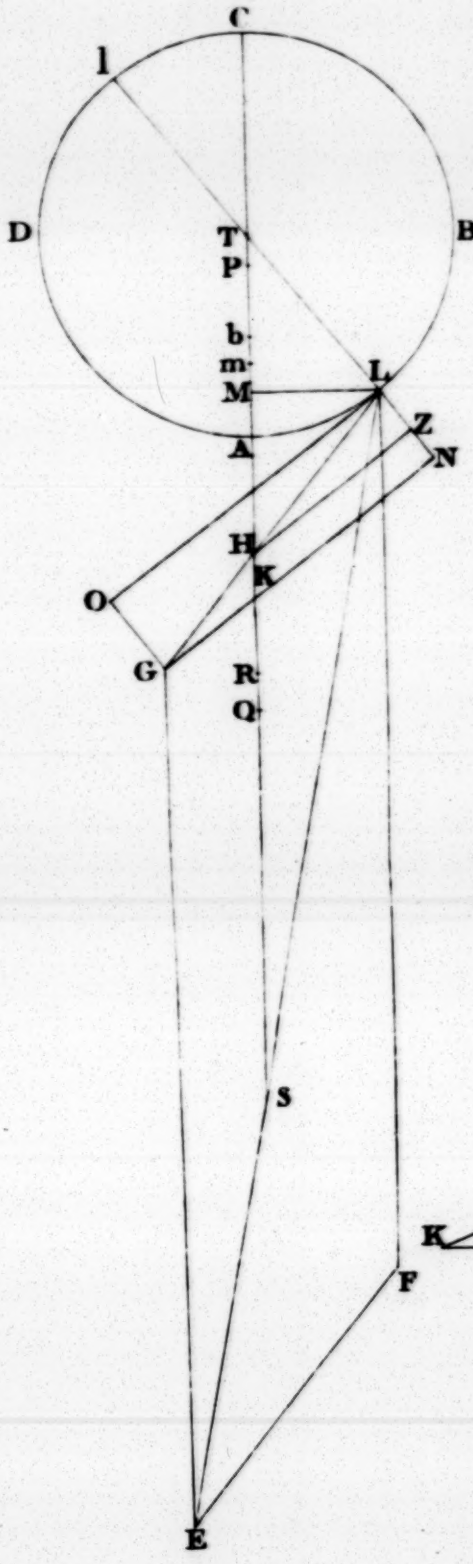

Fig. 2.

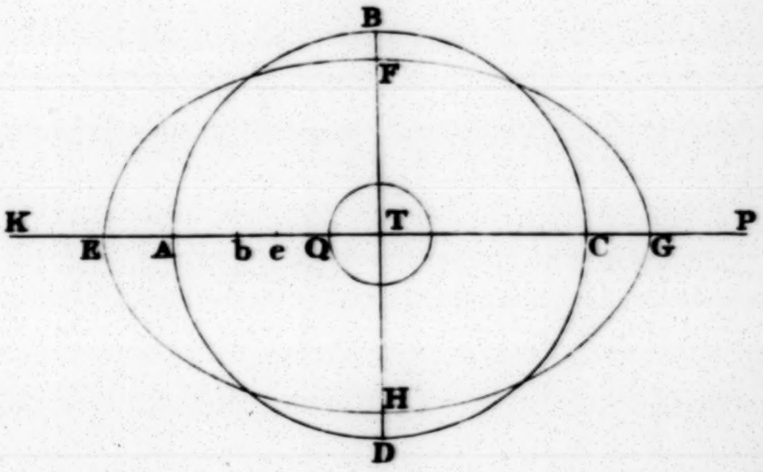

Fig. 3.